SOLUTIONS MANUAL TO ACCOMPANY

ANTENNA THEORY AND DESIGN

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Solutions Manual

to accompany

ANTENNA THEORY AND DESIGN

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CHAPTER 1

<u>1.2-1</u> Using
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (1\text{-}1)$$
with (1-6)
$$\nabla \times \left[Re\left(\vec{E} e^{j\omega t}\right)\right] = -\frac{\partial}{\partial t} Re\left[\vec{B} e^{j\omega t}\right]$$
or
$$Re\left[e^{j\omega t} \nabla \times \vec{E}\right] = Re\left[-B j\omega e^{j\omega t}\right]$$
from which
$$\underline{\nabla \times \vec{E} = -j\omega \vec{B}} \qquad (1\text{-}7)$$
If the last step is not obvious one can use the following.
If $Re\left[c_1 e^{j\omega t}\right] = Re\left[c_2 e^{j\omega t}\right]$ $\quad c_1, c_2$ complex nos.
or
$$Re\left[(a_1 + jb_1)(\cos\omega t + j\sin\omega t)\right] = Re\left[(a_2 + jb_2)(\cos\omega t + j\sin\omega t)\right]$$
So
$$a_1 \cos\omega t - b_1 \sin\omega t = a_2 \cos\omega t - b_2 \sin\omega t$$
which must be true for any time t since a_1, b_1, a_2, b_2 are real constants. Thus $a_1 = a_2$ and $b_1 = b_2$. Hence
$$c_1 = c_2$$
Applying this to the above, let $c_1 \rightarrow \nabla \times \vec{E}$ and $c_2 \rightarrow -j\omega \vec{B}$, from which (1-7) follows.

<u>1.2-2</u> Using (1-13) in (1-9) gives
$$\rho_T = \nabla \cdot \vec{D} = \nabla \cdot (\epsilon \vec{E})$$
From (1-11) $\quad \rho_T = -\dfrac{\nabla \cdot \vec{J}_T}{j\omega}$

So
$$\nabla \cdot (\epsilon \vec{E}) = -\frac{\nabla \cdot \vec{J}_T}{j\omega} = -\frac{1}{j\omega} \nabla \cdot (\sigma \vec{E} + \vec{J}) = \nabla \cdot \left(\frac{-\sigma}{j\omega} \vec{E}\right) - \frac{1}{j\omega} \nabla \cdot \vec{J}$$
Or
$$\nabla \cdot \left(\epsilon \vec{E} + \frac{\sigma}{j\omega} \vec{E}\right) = \nabla \cdot (\epsilon' \vec{E}) = -\frac{1}{j\omega} \nabla \cdot \vec{J}$$
But $\quad \nabla \cdot \vec{J} = -j\omega \rho \qquad (1\text{-}20)$

Hence $\quad \nabla \cdot (\epsilon' \vec{E}) = \rho \quad$ which is (1-18)

Note $-j\omega \rho_T = \nabla \cdot \vec{J}_T = \nabla \cdot (\sigma \vec{E}) + \nabla \cdot \vec{J} = \nabla \cdot (\sigma \vec{E}) - j\omega \rho \Rightarrow \rho_T = \dfrac{\nabla \cdot (\sigma \vec{E})}{-j\omega} + \rho$

1.2-3 From (1-16) and (1-17)

$$\nabla \times \vec{E} = -j\omega\mu \vec{H}$$

$$\nabla \times \vec{H}^* = \left[j\omega(\epsilon - j\tfrac{\sigma}{\omega})\vec{E} + \vec{J}\right]^* = -j\omega\epsilon\vec{E}^* + \sigma\vec{E}^* + \vec{J}^*$$

From (A-19)

$$\nabla\cdot(\vec{E}\times\vec{H}^*) = \vec{H}^*\cdot(\nabla\times\vec{E}) - \vec{E}\cdot\nabla\times\vec{H}^*$$

$$= \vec{H}^*\cdot(-j\omega\mu\vec{H}) - \vec{E}\cdot(-j\omega\epsilon\vec{E}^* + \sigma\vec{E}^* + \vec{J}^*)$$

$$= -j\omega\mu|H|^2 + j\omega\epsilon|E|^2 - \sigma|E|^2 - \vec{E}\cdot\vec{J}^*$$

So

$$-\nabla\cdot(\vec{E}\times\vec{H}^*) = \vec{E}\cdot\vec{J}^* + \sigma|E|^2 + j\omega\left(\mu|H|^2 - \epsilon|E|^2\right)$$

Taking $\frac{1}{2}\iiint\ dv$ of both sides, $-\frac{1}{2}\iiint_v \nabla\cdot(\vec{E}\times\vec{H}^*)dv$

$$= -\frac{1}{2}\oint\!\!\oint_S \vec{E}\times\vec{H}^*\cdot d\vec{S} \qquad \text{by divergence theorem (A-23)}$$

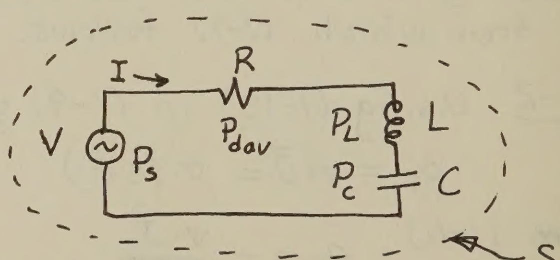
$$= \frac{1}{2}\iiint \vec{E}\cdot\vec{J}^* dv + \frac{1}{2}\iiint \sigma|E|^2 dv + j2\omega\iiint\left[\frac{1}{4}\mu|H|^2 - \frac{1}{4}\epsilon|E|^2\right]dv$$

or

$$-P_f = P_s + P_{d_{av}} + j2\omega(W_{m_{av}} - W_{e_{av}})$$

which is (1-28) and identification of terms in last two equations yields (1-29) to (1-33).

1.2-4



$$V = I(R + j\omega L - j\frac{1}{\omega C})$$

$$P_s = VI^*$$

$$= |I|^2R + j\omega L|I|^2 - j\frac{1}{\omega C}|I|^2$$

$$= 0 + |I|^2R + j2\omega\left(\frac{1}{2}L|I|^2 - \frac{1}{2}\frac{1}{\omega^2 C}|I|^2\right)$$

$$= 0 + P_{d_{av}} + j2\omega(W_{L_{av}} - W_{C_{av}}) \quad \text{which is (1-28) with } P_f = 0$$

where

$$P_{d_{av}} = |I|^2R, \quad W_{L_{av}} = \frac{1}{2}L|I|^2,$$

$$W_{C_{av}} = \frac{1}{2}C|V_c|^2 = \frac{1}{2}C\frac{1}{\omega^2C^2}|I| \qquad \text{since } |V_c| = \frac{1}{\omega C}|I|$$

$$= \frac{1}{2}\frac{1}{\omega^2 C}|I|$$

Note: In circuits $I = \frac{1}{\sqrt{2}}I_{max}e^{j\omega t}$ and $i(t) = Re(I_{max}e^{j\omega t})$ and in fields $\vec{E} = \vec{E}_{max}e^{j\omega t}$

1.3-1 Eqn. (1-18) is

$$\nabla\cdot\vec{E} = \rho/\epsilon'$$

substituting
$$\vec{E} = -j\omega\mu\vec{A} - \nabla\Phi \qquad (1\text{-}39)$$
into this gives
$$\frac{\rho}{\epsilon'} = \nabla\cdot(-j\omega\mu\vec{A} - \nabla\Phi) = -j\omega\mu\,\nabla\cdot\vec{A} - \nabla^2\Phi$$
Substituting $\nabla\cdot\vec{A} = -j\omega\epsilon'\Phi$ (1-44) gives
$$-j\omega\mu(-j\omega\epsilon'\Phi) - \nabla^2\Phi = \rho/\epsilon'$$
or
$$\nabla^2\Phi - \omega^2\mu\epsilon'\Phi = -\rho/\epsilon' \quad \text{which is } (1\text{-}47).$$

1.3-2 (a) The Laplacian in spherical coordinates from (A-36) is
$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \cdots$$
Then, with
$$\psi = C\,e^{-j\beta r}/r \qquad \text{we have}$$
$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 C\frac{\partial}{\partial r}\left(\frac{e^{-j\beta r}}{r}\right)\right) = \frac{C}{r^2}\frac{\partial}{\partial r}\left[r^2\left(-j\beta\frac{e^{-j\beta r}}{r} - \frac{e^{-j\beta r}}{r^2}\right)\right]$$
$$= \frac{C}{r^2}\left[-\beta^2 e^{-j\beta r} r - j\beta e^{-j\beta r} + j\beta e^{-j\beta r}\right]$$
$$= -\beta^2 C\frac{e^{-j\beta r}}{r}$$
So
$$\nabla^2\psi + \beta^2\psi = -\beta^2 C\frac{e^{-j\beta r}}{r} + \beta^2\left(C\frac{e^{-j\beta r}}{r}\right) = 0 \quad \text{verifying } (1\text{-}52).$$

(b) Integrating (1-51) over a small volume v
$$\iiint_v \nabla^2\psi\,dv - \beta^2\iiint_v \psi\,dv = -\iiint_v \delta(x)\delta(y)\delta(z)\,dv = -1$$
And
$$\iiint_v \psi\,dv = C\iiint_v \frac{e^{-j\beta r}}{r} r^2\sin\theta\,d\theta\,d\phi\,dr$$
$$= C\iiint_v e^{-j\beta r}\, r\sin\theta\,d\theta\,d\phi\,dr \xrightarrow[r\to 0]{} 0$$
Using the divergence theorem of (A-23)
$$\iiint_v \nabla^2\psi\,dv = \iiint_v \nabla\cdot\nabla\psi\,dv \equiv \oiint_S \nabla\psi\cdot d\vec{s}$$
$$= \oiint_S \nabla\left(C\frac{e^{-j\beta r}}{r}\right)\cdot\hat{r}\, r^2\sin\theta\,d\theta\,d\phi$$
$$= \oiint_S \hat{r}\,C\left(-j\beta\frac{e^{-j\beta r}}{r} - \frac{e^{-j\beta r}}{r^2}\right)\cdot\hat{r}\,r^2\sin\theta\,d\theta\,d\phi$$

1.3-2b (con't)

$$\iiint_{v} \nabla^2\psi \, dv \xrightarrow[r\to 0]{} -C \oiint_{S} \sin\theta \, d\theta \, d\phi = -C\,4\pi$$

So substituting into the first eqn.

$$-C4\pi + 0 = -1 \Rightarrow \boxed{C = \frac{1}{4\pi}}$$

1.4-1

From (1-67)

$$E_\theta = \frac{I\Delta z}{4\pi} j\omega\mu \left(1 + \frac{\sqrt{\frac{\mu}{\epsilon}}}{j\omega\mu}\frac{1}{r} + \frac{1}{j\omega\epsilon j\omega\mu}\frac{1}{r^2}\right)\frac{e^{-j\beta r}}{r}\sin\theta$$

$$E_r = \frac{I\Delta z}{2\pi} j\omega\mu \left(\frac{\sqrt{\frac{\mu}{\epsilon}}}{j\omega\mu}\frac{1}{r} + \frac{1}{j\omega\epsilon j\omega\mu}\frac{1}{r^2}\right)\frac{e^{-j\beta r}}{r}\sin\theta$$

But

$$\frac{\sqrt{\frac{\mu}{\epsilon}}}{j\omega\mu} = \frac{1}{j\omega\sqrt{\mu\epsilon}} = \frac{1}{j\beta} \quad \& \quad \frac{1}{j\omega\epsilon j\omega\mu} = \frac{1}{j^2\omega^2\mu\epsilon} = \frac{1}{(j\omega\sqrt{\mu\epsilon})^2} = \frac{1}{(j\beta)^2}$$

So

$$E_\theta = \frac{I\Delta z}{4\pi} j\omega\mu \left(1 + \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2}\right)\frac{e^{-j\beta r}}{r}\sin\theta$$

$$E_r = \frac{I\Delta z}{2\pi} j\omega\mu \left(\frac{1}{j\beta r} + \frac{1}{(j\beta r)^2}\right)\frac{e^{-j\beta r}}{r}\sin\theta$$

from which (1-70) follows directly.

1.4-2

(a) From (1-58)

$$\vec{E} = \frac{1}{j\omega\epsilon}\nabla\times\vec{H}$$

From (1-69)

$$\vec{H} = \frac{I\Delta z}{4\pi} j\beta\left(1 + \frac{1}{j\beta r}\right)\frac{e^{-j\beta r}}{r}\sin\theta\,\hat{\phi}$$

First, from (A-35),

$$\nabla\times\vec{H} = \hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(H_\phi\sin\theta) - \hat{\theta}\frac{1}{r}\frac{\partial}{\partial r}(r H_\phi)$$

$$= \hat{r}\frac{1}{r\sin\theta}\frac{I\Delta z}{4\pi} j\beta\left(1+\frac{1}{j\beta r}\right)\frac{\partial}{\partial\theta}(\sin^2\theta)\frac{e^{-j\beta r}}{r}$$

$$-\hat{\theta}\frac{1}{r}\frac{I\Delta z}{4\pi} j\beta\frac{\partial}{\partial r}\left[\left(1+\frac{1}{j\beta r}\right)e^{-j\beta r}\right]\sin\theta$$

$$= \hat{r}\frac{1}{r\sin\theta}\frac{I\Delta z}{4\pi} j\beta\left(1+\frac{1}{j\beta r}\right)\frac{e^{-j\beta r}}{r}2\sin\theta\cos\theta$$

$$-\hat{\theta}\frac{I\Delta z}{4\pi} j\beta\left[\frac{-1}{j\beta r^2} + \left(1+\frac{1}{j\beta r}\right)(-j\beta)\right]e^{-j\beta r}\sin\theta$$

1.4-2a (con't)

$$\nabla\times\vec{H} = \hat{r}\ \frac{I\Delta z}{2\pi}(j\beta)^2\left[\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\right]\frac{e^{-j\beta r}}{r}\cos\theta$$

$$-\hat{\theta}\ \frac{I\Delta z}{4\pi}j\beta(-j\beta)\left[1+\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\right]\frac{e^{-j\beta r}}{r}\sin\theta$$

Then

$$\vec{E}=\frac{1}{j\omega\epsilon}\nabla\times\vec{H} = \hat{r}\ \frac{I\Delta z}{2\pi}\ \frac{j^2\omega^2\mu\epsilon}{j\omega\epsilon}\left[\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\right]\frac{e^{-j\beta r}}{r}\cos\theta$$

$$-\hat{\theta}\ \frac{I\Delta z}{4\pi}\ \frac{-j^2\omega^2\mu\epsilon}{j\omega\epsilon}\left[1+\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\right]\frac{e^{-j\beta r}}{r}\sin\theta$$

$$=\hat{\theta}\ \frac{I\Delta z}{4\pi}j\omega\mu\left[1+\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\right]\frac{e^{-j\beta r}}{r}\sin\theta$$

$$+\hat{r}\ \frac{I\Delta z}{2\pi}j\omega\epsilon\left[\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\right]\frac{e^{-j\beta r}}{r}\cos\theta$$

which is (1-70)

(b)

Using $\epsilon'=\epsilon$ in (1-46):

$$\vec{E}=-j\omega\mu\vec{A}+\frac{\nabla(\nabla\cdot\vec{A})}{j\omega\epsilon}$$

And (1-60) with (A-3) is

$$\vec{A}=\frac{I e^{-j\beta r}}{4\pi r}\Delta z\,\hat{z}=\frac{I\Delta z\, e^{-j\beta r}}{4\pi r}(\hat{r}\cos\theta-\hat{\theta}\sin\theta)$$

Now (A-34) leads to

$$\nabla\cdot\vec{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta)+\frac{1}{r\sin\theta}\xcancel{\frac{\partial A_\phi}{\partial\phi}}^{\,0}$$

$$=\frac{I\Delta z}{4\pi}\left[\frac{\cos\theta}{r^2}\frac{\partial}{\partial r}(re^{-j\beta r})-\frac{e^{-j\beta r}}{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)\right]$$

$$=\frac{I\Delta z}{4\pi}\left[\frac{\cos\theta}{r^2}(e^{-j\beta r}-j\beta re^{-j\beta r})-\frac{e^{-j\beta r}}{r^2\sin\theta}2\sin\theta\cos\theta\right]$$

$$=-\frac{I\Delta z}{4\pi}\cos\theta\,\frac{e^{-j\beta r}}{r^2}(1+j\beta r)$$

And from (A-33)

$$\nabla(\nabla\cdot\vec{A})=\left[\hat{r}\frac{\partial}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right]\nabla\cdot\vec{A}$$

$$=-\frac{I\Delta z}{4\pi}\left[\hat{r}\cos\theta\frac{\partial}{\partial r}\left(e^{-j\beta r}\left(\frac{1}{r^2}+\frac{j\beta}{r}\right)\right)+\hat{\theta}\frac{e^{-j\beta r}}{r^3}(1+j\beta r)\frac{\partial}{\partial\theta}\cos\theta\right]$$

1.4-2b (con't)

$$\nabla(\nabla\cdot\vec{A}) = -\frac{I\Delta z}{4\pi}\Big[\hat{r}\cos\theta\Big(-j\beta e^{-j\beta r}\Big(\frac{1}{r^2}+\frac{j\beta}{r}\Big)+e^{-j\beta r}\Big(-\frac{2}{r^3}-\frac{j\beta}{r^2}\Big)\Big)$$
$$+\hat{\theta}\,\frac{e^{-j\beta r}}{r^3}(1+j\beta r)(-\sin\theta)\Big]$$
$$= \frac{I\Delta z}{4\pi}\frac{e^{-j\beta r}}{r^2}\Big[\hat{r}\cos\theta\Big(j\beta-\beta^2 r+\frac{2}{r}+j\beta\Big)+\hat{\theta}\sin\theta\Big(\frac{1}{r}+j\beta\Big)\Big]$$

So
$$\vec{E} = -j\omega\mu\vec{A} + \frac{\nabla(\nabla\cdot\vec{A})}{j\omega\epsilon}$$
$$= \frac{I\Delta z}{4\pi}j\omega\mu\frac{e^{-j\beta r}}{r}\Big\{\Big[-\hat{r}\cos\theta+\hat{\theta}\sin\theta\Big]$$
$$+\frac{1}{j\omega\mu j\omega\epsilon}\Big[\hat{r}\cos\theta\Big(-\beta^2+\frac{2j\beta}{r}+\frac{2}{r^2}\Big)+\hat{\theta}\sin\theta\Big(\frac{j\beta}{r}+\frac{1}{r^2}\Big)\Big]\Big\}$$
$$= \frac{I\Delta z}{4\pi}j\omega\mu\frac{e^{-j\beta r}}{r}\Big\{\hat{r}\cos\theta\Big[-1+1+\frac{2}{j\beta r}+\frac{2}{(j\beta r)^2}\Big]$$
$$+\hat{\theta}\sin\theta\Big[1+\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\Big]\Big\}$$

since $j\omega\mu j\omega\epsilon = -\omega^2\mu\epsilon = (j\beta)^2$

Thus
$$\vec{E} = \frac{I\Delta z}{4\pi}j\omega\mu\Big(1+\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\Big)\frac{e^{-j\beta r}}{r}\sin\theta\,\hat{\theta}$$
$$+\frac{I\Delta z}{2\pi}j\omega\mu\Big(\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\Big)\frac{e^{-j\beta r}}{r}\cos\theta\,\hat{r}$$
which is (1-70)

1.4-3

(a) The complex Poynting vector is

$$\frac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{2}(E_\theta\hat{\theta}+E_r\hat{r})\times H_\phi^*\hat{\phi} = \frac{1}{2}(E_\theta H_\phi^*\hat{r}-E_rH_\phi^*\hat{\theta})$$
$$= \frac{1}{2}\Big(\frac{I\Delta z}{4\pi}\Big)^2\Big\{j\omega\mu\Big(1+\frac{1}{j\beta r}-\frac{1}{\beta^2r^2}\Big)\frac{e^{-j\beta r}}{r}\sin\theta\cdot(-j\beta)\Big(1-\frac{1}{j\beta r}\Big)\frac{e^{+j\beta r}}{r}\sin\theta\,\hat{r}$$
$$-2j\omega\mu\Big(\frac{1}{j\beta r}-\frac{1}{\beta^2r^2}\Big)\frac{e^{-j\beta r}}{r}\cos\theta\cdot(-j\beta)\Big(1-\frac{1}{j\beta r}\Big)\frac{e^{+j\beta r}}{r}\sin\theta\,\hat{\theta}\Big\}$$

where (1-69) and (1-70) were used.

$$= \frac{1}{2}\Big(\frac{I\Delta z}{4\pi}\Big)^2\Big\{j\omega\mu(-j\beta)\frac{1}{r^2}\Big[1+\cancel{\frac{1}{j\beta r}}-\cancel{\frac{1}{\beta^2r^2}}-\cancel{\frac{1}{j\beta r}}+\cancel{\frac{1}{\beta^2r^2}}+\frac{1}{j\beta^3r^3}\Big]$$
$$\cdot\sin^2\theta\,\hat{r}-2j\omega\mu(-j\beta)\frac{1}{r^2}\Big[\frac{1}{j\beta r}-\cancel{\frac{1}{\beta^2r^2}}+\cancel{\frac{1}{\beta^2r^2}}+\frac{1}{j\beta^3r^3}\Big]$$
$$\cdot\cos\theta\sin\theta\,\hat{\theta}\Big\}$$

1.4-3a (con't)

Then $\frac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{2}\left(\frac{I\Delta z}{4\pi}\right)^2 \beta\omega\mu \frac{1}{r^2}\Big[\left(1 - j\frac{1}{\beta^3 r^3}\right)\sin^2\theta\,\hat{r}$

$$+2\left(j\frac{1}{\beta r} + j\frac{1}{\beta^3 r^3}\right)\cos\theta\sin\theta\,\hat{\theta}\Big]$$

(b)

$$P_{av} = \mathrm{Re}\Big[\iint_{sphere} \tfrac{1}{2}\vec{E}\times\vec{H}^*\cdot d\vec{S}\Big] = \iint_{sphere} \mathrm{Re}\left(\tfrac{1}{2}\vec{E}\times\vec{H}^*\right)\cdot d\vec{S}$$

Taking the real part of the answer in (a)

$$P_{av} = \iint_{sphere} \frac{1}{2}\left(\frac{I\Delta z}{4\pi}\right)^2 \beta\omega\mu \frac{1}{r^2}\sin^2\theta\;\hat{r}\cdot d\vec{S}$$

$$= \frac{1}{2}\left(\frac{I\Delta z}{4\pi}\right)^2 \beta\omega\mu \int_0^{2\pi}\int_0^{\pi} \frac{\sin^2\theta}{r^2}\hat{r}\cdot\hat{r}\; r^2\sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{2}\left(\frac{I\Delta z}{4\pi}\right)^2 \beta\omega\mu \underbrace{\int_0^{2\pi} d\phi}_{2\pi}\;\underbrace{\int_0^{\pi}\sin^3\theta\, d\theta}_{4/3}$$

since

$$\int_0^{\pi}\sin^3\theta\, d\theta = \frac{1}{4}\int_0^{\pi}(3\sin\theta - \sin 3\theta)\,d\theta = \frac{1}{4}\Big[-3\cos\theta + \frac{\cos 3\theta}{3}\Big]_0^{\pi}$$

$$= \frac{1}{4}\Big[-3(-1) + \frac{-1}{3} - (-3)(+1) - \frac{1}{3}\Big] = \frac{1}{4}\Big[3 - \frac{1}{3} + 3 - \frac{1}{3}\Big] = \frac{1}{4}\frac{16}{3} = \frac{4}{3}$$

$$= \frac{1}{2}\left(\frac{I\Delta z}{4\pi}\right)^2 \beta\omega\mu\, 2\pi\frac{4}{3} = \underline{\frac{(I\Delta z)^2}{12\pi}\beta\omega\mu}$$ which is (1-74).

(c) The reason that P_{av} equals the power radiated is that the radiated power is present for all values of r (distance from dipole). For small values of r it is accompanied by reactive power components as demonstrated in (a), but the real part (radiated power) expression is the same. In other words, the total radiated power is independent of r.

<u>1.4-4</u>

From (A-34)

$$\nabla\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\theta\sin\theta) + \frac{1}{r\sin\theta}\cancel{\frac{\partial E_\phi}{\partial\phi}}^{\,0}$$

since $E_\phi = 0$. From (1-70):

1.4-4 (con't)

$$E_\theta = \frac{I\Delta z}{4\pi} j\omega\mu\left(1+\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\right)\frac{e^{-j\beta r}}{r}\sin\theta$$

$$E_r = \frac{I\Delta z}{2\pi} j\omega\mu\left(\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\right)\frac{e^{-j\beta r}}{r}\cos\theta$$

Consider

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = \frac{I\Delta z}{2\pi} j\omega\mu \cos\theta \frac{1}{r^2}\frac{\partial}{\partial r}\left[\left(\frac{1}{j\beta}+\frac{1}{(j\beta)^2 r}\right)e^{-j\beta r}\right]$$

$$= \frac{I\Delta z}{2\pi} j\omega\mu \cos\theta \frac{1}{r^2}\left[-\frac{1}{(j\beta)^2 r^2}+\left(\frac{1}{j\beta}+\frac{1}{(j\beta)^2 r^2}\right)(-j\beta)\right]e^{-j\beta r}$$

$$= \frac{I\Delta z}{2\pi} j\omega\mu \cos\theta\left[\frac{1}{\beta^2 r^3}-\frac{1}{r}+j\frac{1}{\beta r^2}\right]\frac{e^{-j\beta r}}{r} \qquad (a)$$

And

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\theta\sin\theta) = \frac{1}{r\sin\theta}\ \frac{I\Delta z}{4\pi} j\omega\mu\left(1-j\frac{1}{\beta r}-\frac{1}{\beta^2 r^2}\right)\frac{e^{-j\beta r}}{r}\frac{\partial}{\partial\theta}(\sin^2\theta)$$

$$= \frac{I\Delta z}{4\pi} j\omega\mu\, 2\cos\theta\left(\frac{1}{r}-j\frac{1}{\beta r^2}-\frac{1}{\beta^2 r^3}\right)\frac{e^{-j\beta r}}{r} \qquad (b)$$

The sum of (a) and (b) is zero, thus

$$\nabla\cdot\vec{E} = 0. \qquad \text{QED.}$$

<u>1.5-1</u>

Eqn. (1-84) is $\quad A_z = \dfrac{e^{-j\beta r}}{4\pi r}\displaystyle\int I(z')e^{j\beta z'\cos\theta}dz'$

And (1-89) is

$$\vec{E} = -j\omega\mu\vec{A} + \frac{\nabla(\nabla\cdot\vec{A})}{j\omega\epsilon}$$

First

$$\nabla\cdot\vec{A} = \nabla\cdot(-A_z\sin\theta\,\hat{\theta} + A_z\cos\theta\,\hat{r})$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \qquad \text{from (A-34)}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cos\theta\frac{e^{-j\beta r}}{4\pi r}\int I(z')e^{j\beta z'\cos\theta}dz'\right)$$

$$+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{-\sin\theta\, e^{-j\beta r}}{4\pi r}\int I(z')e^{j\beta z'\cos\theta}dz'\right) + 0$$

$$= \cos\theta\int I(z')e^{j\beta z'\cos\theta}dz'\,\frac{1}{r^2}\left(\frac{r(-j\beta)e^{-j\beta r}}{4\pi}+\frac{e^{-j\beta r}}{4\pi}\right)$$

$$+\frac{-1}{\sin\theta}\,\frac{e^{-j\beta r}}{4\pi r^2}\frac{\partial}{\partial\theta}\left(\sin\theta\int I(z')e^{j\beta z'\cos\theta}dz'\right)$$

1.5-1 (con't)

The second term may be neglected since it varies as $1/r^2$ and the subsequent gradient operation will lead to even faster fall off with r, i.e. $1/r^3$.

Then
$$\nabla\cdot\vec{A} \approx \cos\theta \int I(z')e^{j\beta z'\cos\theta}dz' \,\frac{e^{-j\beta r}}{4\pi}\left(\frac{-j\beta r}{r}+\frac{1}{r^2}\right)$$

For the same reason we can neglect the second term above. (Note $\frac{1/r^2}{-j\beta/r}=\frac{1}{j\beta r}$ which is $\ll 1$ if $\beta r \gg 1$.)

So
$$\nabla\cdot\vec{A} \approx -j\beta\cos\theta\,\frac{e^{-j\beta r}}{4\pi r}\int I(z')e^{j\beta z'\cos\theta}dz' = -j\beta\cos\theta\, A_z$$

From (A-33)
$$\nabla(\nabla\cdot\vec{A}) = \hat{r}\frac{\partial}{\partial r}\nabla\cdot\vec{A} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\nabla\cdot\vec{A} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\nabla\cdot\vec{A}$$
$$= \hat{r}\frac{\partial}{\partial r}\left(-j\beta\cos\theta\,\frac{e^{-j\beta r}}{4\pi r}\int I(z')e^{j\beta z'\cos\theta}dz'\right.$$
$$\left.+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(-j\beta\cos\theta\,\frac{e^{-j\beta r}}{4\pi r}\int I(z')e^{j\beta z'\cos\theta}dz'\right) + 0\right.$$
$$= -j\beta\cos\theta\int I(z')e^{j\beta z'\cos\theta}dz'\left(\frac{-j\beta e^{-j\beta r}}{4\pi r} - \frac{e^{-j\beta r}}{4\pi r^2}\right)\hat{r}$$
$$+(-j\beta)\frac{e^{-j\beta r}}{4\pi r^2}\frac{\partial}{\partial\theta}\left(\cos\theta\int I(z')e^{j\beta z'\cos\theta}dz'\right)\hat{\theta}$$
$$\approx -j\beta\cos\theta\,(-j\beta)\frac{e^{-j\beta r}}{4\pi r}\int I(z')e^{j\beta z'\cos\theta}dz'\,\hat{r} \qquad \text{neglecting } 1/r^2 \text{ terms}$$
$$= -\beta^2\cos\theta\, A_z\,\hat{r}$$

So
$$\vec{E} = -j\omega\mu\vec{A} + \frac{\nabla(\nabla\cdot\vec{A})}{j\omega\epsilon} = -j\omega\mu(-A_z\sin\theta\,\hat{\theta} + A_z\cos\theta\,\hat{r}) + \frac{-\beta^2\cos\theta A_z}{j\omega\epsilon}\hat{r}$$
$$= j\omega\mu A_z\sin\theta\,\hat{\theta} - A_z\left(j\omega\mu + \frac{\beta^2}{j\omega\epsilon}\right)\cos\theta\,\hat{r}$$
$$= j\omega\mu A_z\sin\theta\,\hat{\theta} - A_z\cos\theta\left(j\omega\mu - j\frac{\omega^{\cancel{2}}\mu\cancel{\epsilon}}{\cancel{\omega\epsilon}}\right)\hat{r}$$
$$= j\omega\mu A_z\sin\theta\,\hat{\theta} \quad \text{which is (1-90).}$$

<u>1.5-2</u>

See Section 4.1 for solution of the uniform line source.

1.5-3

$$R = |\vec{r} - \vec{r}'| = \left[(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')\right]^{1/2} = \left[\vec{r}\cdot\vec{r} - 2\vec{r}\cdot\vec{r}' + \vec{r}'\cdot\vec{r}'\right]^{1/2}$$

$$= \left[r^2 - 2\vec{r}\cdot\vec{r}' + (r')^2\right]^{1/2} = r\left[1 - 2\frac{\vec{r}\cdot\vec{r}'}{r^2} + \left(\frac{r'}{r}\right)^2\right]^{1/2} \approx r\left[1 - 2\frac{\vec{r}\cdot\vec{r}'}{r^2}\right]^{\frac{1}{2}}$$

$$\approx r\left[1 - \frac{1}{2}2\frac{\hat{r}\cdot\vec{r}'}{r}\right] = \underline{r - \hat{r}\cdot\vec{r}'} \qquad (1\text{-}93) \qquad r' \ll r$$

1.5-4

For linear antennas $D=L$ in (1-96) and (1-97)

L	$r_{ff} = \frac{2L^2}{\lambda}$	Is $r_{ff} \gg L$?	Is $r_{ff} \gg \lambda$?	Is r_{ff} in column 2 valid?
5λ	50λ	Yes	Yes	Yes
$\lambda/2$	$\lambda/2$	No	No	No
$\lambda/100$	0.0002λ	No	No	No

1.5-5

The far-field region is for

$r > 5D$

$r > 1.6\lambda$

$r > \frac{2D^2}{\lambda}$

The boundaries are then

$r = 5D$

$r = 1.6\lambda$

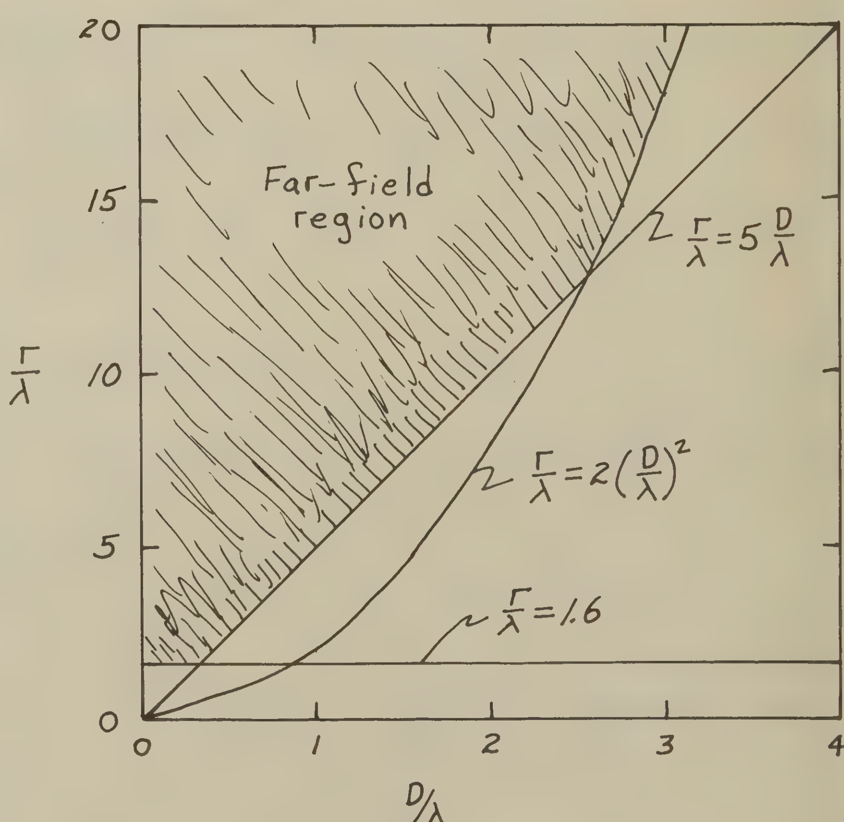
$r = \frac{2D^2}{\lambda}$

or

$\frac{r}{\lambda} = 5\frac{D}{\lambda}$

$\frac{r}{\lambda} = 1.6$

$\frac{r}{\lambda} = 2\left(\frac{D}{\lambda}\right)^2$



1.5-6 $\quad D/\lambda = \dfrac{1\,m}{300\,m} = 0.0033$

So from the graph of Prob. 1.5-5, $\frac{r_{ff}}{\lambda} = 1.6$ applies and

$$r_{ff} = 1.6\lambda = 1.6(300) = 480\,m$$

1.6-1

$$\iint d\Omega = \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta d\phi = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta = 2\pi[-\cos\theta]_0^{\pi}$$

$$= 2\pi[-(-1)-(-1)] = \underline{\underline{4\pi}}$$

1.6-2

(a) From (1-140)

$$\Omega_A = \iint |F(\theta,\phi)|^2 d\Omega = \iint P(\theta) d\Omega = \int_0^{2\pi}\int_0^{\pi} |\cos^n\theta| \sin\theta\, d\theta d\phi$$

$$= 2\pi(2)\int_0^{\pi/2} \cos^n\theta \sin\theta\, d\theta$$

For $n=1$

$$\Omega_A = 4\pi\int_0^{\pi/2} \cos\theta\sin\theta\, d\theta = 4\pi\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} = 2\pi[1-0] = 2\pi$$

For $n=2$

$$\Omega_A = 4\pi\int_0^{\pi/2} \cos^2\theta\sin\theta\, d\theta = 4\pi\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2} = \frac{4\pi}{3}$$

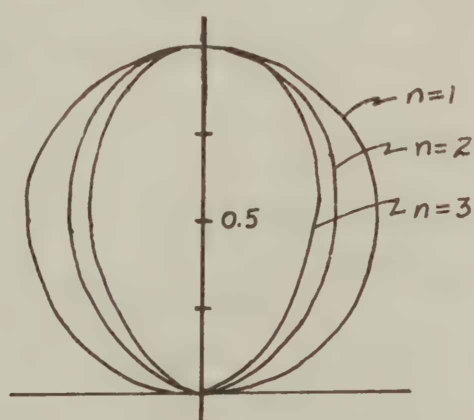
For $n=3$

$$\Omega_A = 4\pi\int_0^{\pi/2} \cos^3\theta\sin\theta\, d\theta = 4\pi\left[-\frac{\cos^4\theta}{4}\right]_0^{\pi/2} = \pi$$

So

n	$D = 4\pi/\Omega_A$
1	2
2	3
3	4

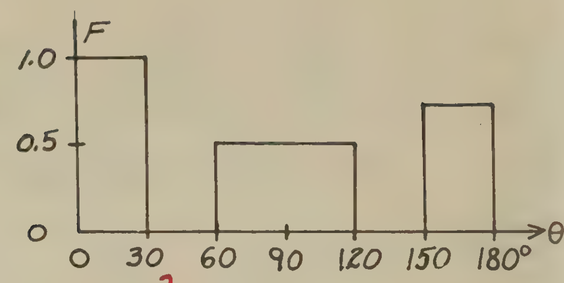
(b)



(c) For $n=0$ the power pattern is $P(\theta) = |\cos^0\theta| = 1$, which is an <u>isotropic</u> pattern, so the directivity is unity.
To prove this $\Omega_A = \int_0^{2\pi}\int_0^{\pi} 1 \sin\theta\, d\theta d\phi = 2\pi[-\cos\theta]_0^{\pi} = 2\pi(2) = 4\pi$

So $D = 4\pi/\Omega_A = 4\pi/4\pi = 1 \longrightarrow$ isotropic pattern

1.6-3



From (1-140)

$$\Omega_A = \iint |F(\theta,\phi)|^2 \, d\Omega$$
$$= \int_0^{2\pi} d\phi \int_0^{\pi} |F(\theta)|^2 \sin\theta \, d\theta$$
$$= 2\pi \left\{ (1)^2 \int_0^{30^\circ} \sin\theta \, d\theta + (0.5)^2 \int_{60^\circ}^{120^\circ} \sin\theta \, d\theta + (0.707)^2 \int_{150^\circ}^{180^\circ} \sin\theta \, d\theta \right\}$$
$$= 2\pi \left\{ (-\cos 30^\circ + 1) + 0.5(\cos 60^\circ - \cos 120^\circ) + 0.707(1 + \cos 150^\circ) \right\}$$
$$= 2\pi (0.2287) = 1.437$$

So
$$D = \frac{4\pi}{\Omega_A} = \frac{4\pi}{2\pi(0.2287)} = \frac{2}{0.2287} = \boxed{\cancel{8.745}\ 4.43 = 9.42 \text{ dB}}$$

At $\theta = 90^\circ$, from (1-144)
$$D(\theta=90^\circ) = D \cdot |F(\theta=90^\circ)|^2 = 8.745 \cdot (0.5)^2 = \cancel{2.186}\ 1.1 = 3.\cancel{40} \text{ dB}$$

1.6-4

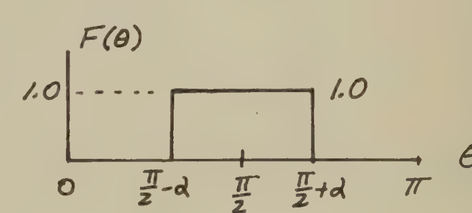
$$D = \frac{4\pi}{\Omega_A} \approx \frac{4\pi}{HP_E \, HP_H} = \frac{4\pi}{\frac{\pi}{180} HP_{E^\circ} \frac{\pi}{180} HP_{H^\circ}} = \frac{\left(\frac{180}{\pi}\right)^2 4\pi}{HP_{E^\circ} HP_{H^\circ}} = \frac{41{,}253}{HP_{E^\circ} HP_{H^\circ}}$$

1.6-5

From Prob. 1.6-4 with $HP_{E^\circ} = HP_{H^\circ} = 29^\circ$

$$D \approx \frac{41{,}253}{(29)^2} = 49.05 = \boxed{16.9 \text{ dB}}$$ using (1-155) $D_{dB} = 10 \log D$

1.6-6



$$\Omega_A = \iint |F(\theta,\phi)|^2 \, d\Omega$$
$$= \int_0^{2\pi} \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} 1^2 \sin\theta \, d\theta \, d\phi$$
$$= (2\pi) \left[-\cos\theta \right]_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha}$$
$$= 2\pi \left[-\cos\left(\tfrac{\pi}{2}+\alpha\right) + \cos\left(\tfrac{\pi}{2}-\alpha\right) \right] = 2\pi(\sin\alpha - \sin(-\alpha)) = 4\pi \sin\alpha$$

$$D = \frac{4\pi}{\Omega_A} = \frac{4\pi}{4\pi \sin\alpha} = \frac{1}{\sin\alpha} = \boxed{\csc\alpha}$$

1.6-7

$$\Omega_A = \int_0^{2\pi}\int_0^{\pi} |F(\theta,\phi)|^2 d\Omega = \int_0^{\phi_0} d\phi \int_{\theta_1}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta} \sin\theta \, d\theta$$

$$= (\phi_0)\int_{\theta_1}^{\frac{\pi}{2}} \csc\theta \, d\theta = \phi_0 \left[\ln|\tan\tfrac{\theta}{2}|\right]_{\theta_1}^{\pi/2}$$

$$= \phi_0\left[\ln(\tan\tfrac{\pi}{4}) - \ln(\tan\tfrac{\theta_1}{2})\right] = \phi_0\left(0 - \ln(\tan\tfrac{\theta_1}{2})\right) = \phi_0 \ln(\cot\tfrac{\theta_1}{2})$$

$$D = \frac{4\pi}{\Omega_A} = \boxed{\frac{4\pi}{\phi_0 \ln(\cot\frac{\theta_1}{2})}}$$ or it can be shown that $$D = \frac{4\pi}{\phi_0 \ln(\csc\theta_1 + \cot\theta_1)}$$

1.6-8

From (1-153)

$$G = eD = (0.9)(20) = 18 = \boxed{12.55 \text{ dB}}$$ using (1-154) $G_{dB} = 10\log G$

1.7-1

(a) $\nabla\times\vec{E}_a = -j\omega\mu\vec{H}_a - \vec{M}_a$ (1) $\qquad \nabla\times\vec{E}_b = -j\omega\mu\vec{H}_b - \vec{M}_b$ (3)

$\nabla\times\vec{H}_a = j\omega\epsilon\vec{E}_a + \vec{J}_a$ (2) $\qquad \nabla\times\vec{H}_b = j\omega\epsilon\vec{E}_b + \vec{J}_b$ (4)

Take $\vec{H}_b\cdot$ of (1) and $\vec{E}_a\cdot$ of (4) and subtract

$$\vec{H}_b\cdot\nabla\times\vec{E}_a - \vec{E}_a\cdot\nabla\times\vec{H}_b = -j\omega\mu\vec{H}_b\cdot\vec{H}_a - \vec{H}_b\cdot\vec{M}_a - j\omega\epsilon\vec{E}_a\cdot\vec{E}_b - \vec{E}_a\cdot\vec{J}_b$$

$$\equiv \nabla\cdot(\vec{E}_a\times\vec{H}_b) \text{ from (A-19)} \qquad (5)$$

Take $\vec{E}_b\cdot$ of (2) and $\vec{H}_a\cdot$ of (3) and subtract

$$\vec{E}_b\cdot\nabla\times\vec{H}_a - \vec{H}_a\cdot\nabla\times\vec{E}_b = j\omega\epsilon\vec{E}_b\cdot\vec{E}_a + \vec{E}_b\cdot\vec{J}_a + j\omega\mu\vec{H}_a\cdot\vec{H}_b + \vec{H}_a\cdot\vec{M}_b$$

$$\equiv -\nabla\cdot(\vec{E}_b\times\vec{H}_a) \text{ from (A-19)} \qquad (6)$$

Add (5) and (6)

$$\underline{\nabla\cdot(\vec{E}_a\times\vec{H}_b - \vec{E}_b\times\vec{H}_a) = \vec{E}_b\cdot\vec{J}_a + \vec{H}_a\cdot\vec{M}_b - \vec{H}_b\cdot\vec{M}_a - \vec{E}_a\cdot\vec{J}_b} \qquad (7)$$

(b) Integrating (7) over a volume enclosing both sources

$$\iiint_V \nabla\cdot(\vec{E}_a\times\vec{H}_b - \vec{E}_b\times\vec{H}_a)\cdot d\vec{s} \equiv \oiint_S (\vec{E}_a\times\vec{H}_b - \vec{E}_b\times\vec{H}_a)\cdot d\vec{s} \text{ by (A-23)}$$

$$= \iiint_V [\vec{E}_b\cdot\vec{J}_a + \vec{H}_a\cdot\vec{M}_b - \vec{H}_b\cdot\vec{M}_a - \vec{E}_a\cdot\vec{J}_b]dv \text{ from (7)} \qquad (8)$$

Letting the surface s bounding volume v extend to infinity, the fields at s are in the far field of the sources, which are near the origin of a spherical coordinate system. The spherical (plane) wave relations are from (1-126)

$$E_{\theta a} = \eta H_{\phi a} \qquad E_{\phi a} = -\eta H_{\theta a} \qquad (9)$$
$$E_{\theta b} = \eta H_{\phi b} \qquad E_{\phi b} = -\eta H_{\theta b}$$

1.7-1 (con't)

So $\vec{E}_a \times \vec{H}_b - \vec{E}_b \times \vec{H}_a = (E_{\theta a}\hat{\theta} + E_{\phi a}\hat{\phi}) \times (H_{\theta b}\hat{\theta} + H_{\phi b}\hat{\phi})$

$$- (E_{\theta b}\hat{\theta} + E_{\phi b}\hat{\phi}) \times (H_{\theta a}\hat{\theta} + H_{\phi a}\hat{\phi})$$

$$= [E_{\theta a} H_{\phi b} - E_{\phi a} H_{\theta b} - E_{\theta b} H_{\phi a} + E_{\phi b} H_{\theta a}]\hat{r}$$

$$= \eta [H_{\phi a} H_{\phi b} + H_{\theta a} H_{\theta b} - H_{\phi b} H_{\phi a} - H_{\theta b} H_{\theta a}]\hat{r} \quad \text{from (9)}$$

$$\approx 0 \quad \text{over surface } s \text{ at infinity} \qquad (10)$$

Using (10) in (8)

$$\iiint_V [\vec{E}_b \cdot \vec{J}_a - \vec{H}_b \cdot \vec{M}_a]\, dv = \iiint_V [\vec{H}_a \cdot \vec{M}_b - \vec{E}_a \cdot \vec{J}_b]\, dv$$

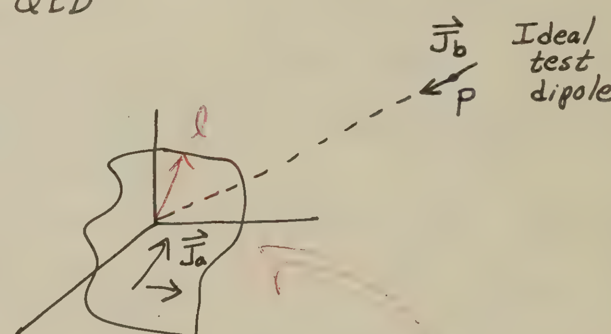
But the LHS (RHS) only has a volume in a volume V_a (V_b) bounding the sources a (b), so

$$\iiint_{V_a} [\vec{E}_b \cdot \vec{J}_a - \vec{H}_b \cdot \vec{M}_a]\, dv = \iiint_{V_b} [\vec{H}_a \cdot \vec{M}_b - \vec{E}_a \cdot \vec{J}_b]\, dv$$

which is (1-159). QED

1.7-2

Choose source b to be an ideal dipole



From (1-160) with $\vec{M}_a = 0$

$$\vec{E}_a(x_p, y_p, z_p) \cdot \vec{p} = \iiint_{V_a} \vec{E}_b \cdot \vec{J}_a\, dv'$$

for finite distribution

and $\vec{p} = \Delta\ell\,\hat{r}$ (ideal dipole), so

$$E_{ar}\,\Delta\ell = \iiint_{V_a} \vec{E}_b \cdot \vec{J}_a\, dv'$$

$\hat{E}_b = I\ell \sin\theta$ [illegible] ≅ [illegible]

But $\vec{E}_b$ is zero at $\vec{J}_a$ since the end on radiation from a short dipole is zero.

Hence E goes as $1/r^2$

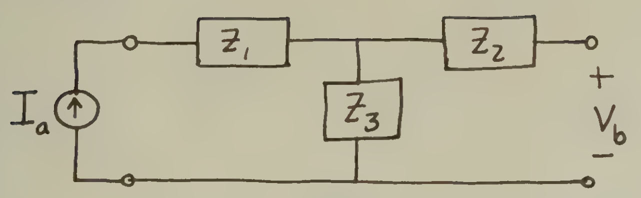
So $\quad E_{ar}\,\Delta\ell = 0$

thus radial components of finite source

∴ $\boxed{E_{ar} = 0}$ for any source a.

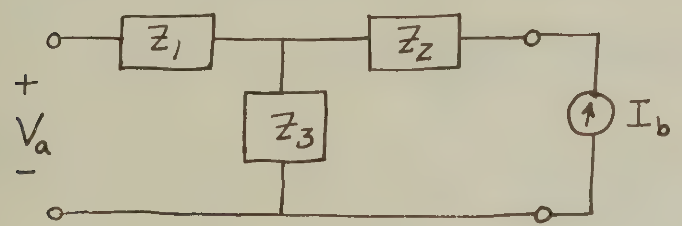
are negligible in far field (not strictly zero)

1.7-3



$$V_b\Big|_{I_b=0} = Z_3 I_a$$

$$Z_{ba} = \frac{V_b}{I_a}\Big|_{I_b=0} = Z_3$$



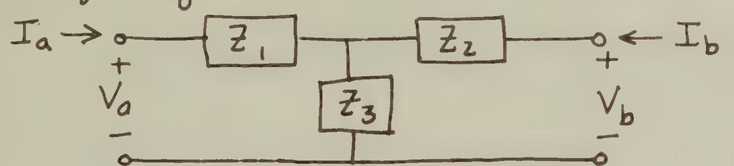
$$V_a\Big|_{I_a=0} = Z_3 I_b$$

$$Z_{ab} = \frac{V_a}{I_b}\Big|_{I_a=0} = Z_3$$

And $Z_{ab} = Z_{ba}$

1.7-4

The voltage equations for the network



are

$$V_a = Z_1 I_a + Z_3(I_a + I_b) = (Z_1 + Z_3) I_a + Z_3 I_b$$

$$V_b = Z_2 I_b + Z_3(I_a + I_b) = Z_3 I_a + (Z_2 + Z_3) I_b$$

Comparing to (1-164) and (1-165),

$$Z_{aa} = Z_1 + Z_3 \qquad Z_{ab} = Z_m = Z_3$$

$$Z_{ba} = Z_m = Z_3 \qquad Z_{bb} = Z_2 + Z_3$$

So

$$Z_1 = Z_{aa} - Z_3 = Z_{aa} - Z_m$$

$$Z_2 = Z_{bb} - Z_3 = Z_{bb} - Z_m$$

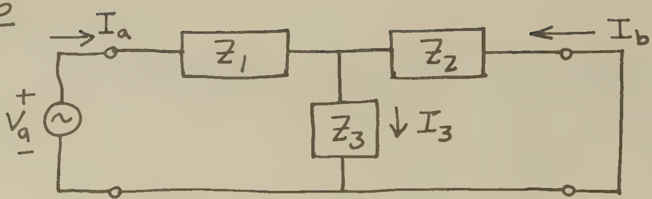
$$Z_3 = Z_m$$

1.7-5

$$Z_{aa} = Z_{bb} \qquad \text{and} \qquad Z_1 = Z_2$$

1.7-6

(a)



$$I_a = I_3 - I_b \qquad I_b = I_3 - I_a \qquad V_3 = -I_b Z_2 = I_3 Z_3$$

So

$$I_b = -\frac{V_3}{Z_2} = -\frac{I_3 Z_3}{Z_2} = -\frac{(I_a + I_b) Z_3}{Z_2}$$

which gives $I_b Z_2 + I_b Z_3 = -I_a Z_3$ or

$$I_b = -I_a \frac{Z_3}{Z_2 + Z_3}$$

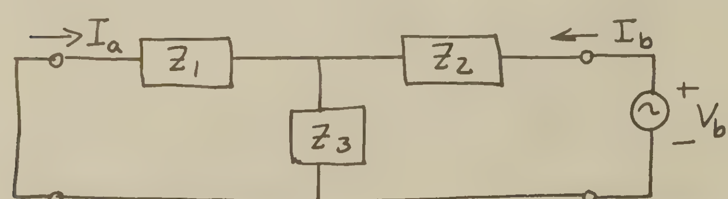
But

$$I_a = \frac{V_a}{Z_1 + \dfrac{Z_2 Z_3}{Z_2 + Z_3}} = \frac{V_a (Z_2 + Z_3)}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}$$

So

$$I_b = -\frac{V_a Z_3}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1} \quad (\text{with } V_b = 0).$$

Now



Similarly $$I_a = -I_b \frac{Z_3}{Z_1 + Z_3}$$

And

$$I_b = \frac{V_b}{Z_2 + \dfrac{Z_1 Z_3}{Z_1 + Z_3}} = \frac{V_b (Z_1 + Z_3)}{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}$$

So

$$I_a = -\frac{V_b Z_3}{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3} \qquad \text{with } V_a = 0$$

Then

$$\left.\frac{V_a}{I_b}\right|_{V_b=0} = \left.\frac{V_b}{I_a}\right|_{V_a=0} = -\frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_3}$$

1.7-6 (con't)

(b) For $V_b = 0$

$$V_a = I_a Z_{aa} + I_b Z_{ab}$$

$$0 = I_a Z_{ba} + I_b Z_{bb} \Rightarrow I_a = -I_b \frac{Z_{bb}}{Z_{ba}}$$

So

$$V_a = -I_b \frac{Z_{bb}}{Z_{ba}} Z_{aa} + I_b Z_{ab}$$

Then

$$\left.\frac{V_a}{I_b}\right|_{V_b=0} = -\frac{Z_{bb}}{Z_{ba}} Z_{aa} + Z_{ab}$$

For $V_a = 0$

$$0 = I_a Z_{aa} + I_b V_{ab} \Rightarrow I_b = -I_a \frac{Z_{aa}}{Z_{ab}}$$

$$V_b = I_a Z_{ba} - I_a \frac{Z_{aa}}{Z_{ab}} Z_{bb}$$

or

$$\left.\frac{V_b}{I_a}\right|_{V_a=0} = Z_{ba} - \frac{Z_{aa} Z_{bb}}{Z_{ab}}$$

Since

$$\left.\frac{V_a}{I_b}\right|_{V_b=0} = \left.\frac{V_b}{I_a}\right|_{V_a=0}$$

$$-\frac{Z_{bb} Z_{aa}}{Z_{ba}} + Z_{ab} = Z_{ba} - \frac{Z_{aa} Z_{bb}}{Z_{ab}} \Rightarrow \boxed{Z_{ab} = Z_{ba} = Z_m}$$

(c) From (a)

$$Z_T = \left.\frac{V_a}{I_b}\right|_{V_b=0} = \left.\frac{V_b}{I_a}\right|_{V_a=0} = -\frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_3}$$

$$= -\frac{Z_1 Z_2}{Z_3} - Z_2 - Z_1$$

$$= -\frac{(Z_{aa} - Z_m)(Z_{bb} - Z_m)}{Z_m} - (Z_{bb} - Z_m) - (Z_{aa} - Z_m)$$

$$= \left(-\frac{Z_{aa} Z_{bb}}{Z_m} + Z_{bb} + Z_{aa} - Z_m\right) - Z_{bb} + Z_m - Z_{aa} + Z_m$$

$$= -\frac{Z_{aa} Z_{bb}}{Z_m} + Z_m \quad \text{which is the answer in (b)}$$

<u>1.8-1</u> $\lambda = \frac{c}{f} = \frac{3 \times 10^8}{5 \times 10^5} = 600 \text{ m} \qquad \frac{\Delta z}{\lambda} = \frac{2}{600} = 3.33 \times 10^{-3} \ll 1$

$$R_s = \sqrt{\frac{\mu \omega}{2\sigma}} = \sqrt{\frac{(4\pi \times 10^{-7})(2\pi \times 5 \times 10^5)}{2(3.5 \times 10^7)}} = 2.375 \times 10^{-4} \,\Omega$$

$$a = \text{radius} = \frac{d}{2} = \frac{6.35 \times 10^{-3}}{2} = 3.175 \times 10^{-3} \text{ m}$$

1.8-1 (con't)

(a) For a uniform current

$R_r = 80\pi^2\left(\frac{\Delta z}{\lambda}\right)^2 = 80\pi^2(3.33\times10^{-3})^2 = 8.77\times10^{-3}\,\Omega$ using (1-180)

$R_{ohmic} = \frac{L}{2\pi a} R_s = \frac{2}{2\pi(3.175\times10^{-3})}(2.375\times10^{-4}) = 2.38\times10^{-2}\,\Omega$ using (1-184)

$$e = \frac{R_r}{R_r + R_{ohmic}} = \frac{8.77\times10^{-3}}{8.77\times10^{-3} + 2.38\times10^{-2}} = 0.269 = \boxed{26.9\%}$$

(b) For a triangular current

$R_r = 20\pi^2\left(\frac{\Delta z}{\lambda}\right)^2 = 20\pi^2(3.33\times10^{-2})^2 = 2.189\times10^{-3}\,\Omega$

$R_{ohmic} = \frac{\Delta z}{2\pi a}\frac{R_s}{3} = \frac{1}{3}R_{ohmic}(\text{uniform}) = \frac{1}{3}(0.0238) = 7.933\times10^{-3}\,\Omega$

using (1-192)

$$e = \frac{R_r}{R_r + R_{ohmic}} = \frac{2.189\times10^{-3}}{2.189\times10^{-3} + 7.933\times10^{-3}} = 0.216 = \boxed{21.6\%}$$

1.8-2

$\lambda = \frac{c}{f} = \frac{3\times10^8}{2.7\times10^7} = 11.1\,m \qquad L = \frac{\lambda}{2} = 5.55\,m \qquad a = \frac{6.35}{2}\,mm = 3.175\times10^{-3}\,m$

$$R_s = \sqrt{\frac{\mu\omega}{2\sigma}} = \sqrt{\frac{4\pi\times10^{-7}\;2\pi\times27\times10^6}{2\times3.5\times10^7}} = 1.745\times10^{-3}\,\Omega$$

$$R_{ohmic} = \frac{L}{2\pi a}\frac{R_s}{3} = \frac{5.55}{2\pi(3.175\times10^{-3})}\frac{1.745\times10^{-3}}{3} = 0.162\,\Omega$$

$$e = \frac{R_r}{R_r + R_{ohmic}} = \frac{70}{70 + 0.162} = \boxed{99.77\%}$$

1.8-3

(a) Uniform current $\qquad I(z) = I_{in} \qquad |z| \le \Delta z/2$

$$P_{ohmic} = \frac{R_s}{2\pi a}\int_{-\frac{\Delta z}{2}}^{\frac{\Delta z}{2}} \frac{1}{2}|I(z)|^2 dz = \frac{R_s}{2\pi a}\frac{1}{2}|I_{in}|^2\int_{-\frac{\Delta z}{2}}^{\frac{\Delta z}{2}} dz$$

$$= \frac{R_s}{2\pi a}\frac{1}{2}|I_{in}|^2 \Delta z$$

$$R_{ohmic} = \frac{2P_{ohmic}}{|I_{in}|^2} = \boxed{\frac{\Delta z}{2\pi a} R_s} \qquad (1\text{-}184)$$

(b) Triangular current $\qquad I(z) = I_{in}\left(1 - \frac{2|z|}{\Delta z}\right) \qquad |z| \le \frac{\Delta z}{2}$

$$P_{ohmic} = \frac{R_s}{2\pi a}\frac{1}{2}|I_{in}|^2\, 2\underbrace{\int_0^{\Delta z/2}\left(1 - \frac{2z}{\Delta z}\right)^2 dz}_{\mathcal{I}}$$

$$\mathcal{I} = \int_0^{\Delta z/2}\left[1 - \frac{4}{\Delta z}z + \frac{4}{(\Delta z)^2}z^2\right]dz = \left[z - \frac{4}{\Delta z}\frac{z^2}{2} + \frac{4}{(\Delta z)^2}\frac{z^3}{3}\right]_0^{\Delta z/2}$$

1.8-3 (con't)

$$\partial = \frac{\Delta z}{2} - 4\frac{1}{2}\frac{1}{4}\Delta z + 4\frac{1}{3}\frac{1}{8}\Delta z = \left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right]\Delta z = \frac{\Delta z}{6}$$

$$P_{ohmic} = \frac{R_s}{2\pi a}|I_{in}|^2 \frac{\Delta z}{6} \;;\; R_{ohmic} = \frac{2P_{ohmic}}{|I_{in}|^2} = \boxed{\frac{\Delta z}{2\pi a}\frac{R_s}{3}} \quad (1-192)$$

<u>1.9-1</u>

(a) For a linear wave $AR = \infty$ and from (1-197)

$$\epsilon = \cot^{-1}(AR) = \cot^{-1}(\infty) = 0$$

From (1-202) $\delta = \tan^{-1}\left(\frac{\tan 2\epsilon}{\sin 2\tau}\right) = \tan^{-1}(0) = 0 \quad \boxed{\delta = 0}$

(b) For circular polarization $|AR| = 1$ and $\text{sign}(AR) = -1$ for right hand sense. So $AR = -1$

Then $\epsilon = \cot^{-1}(-1) = -45°$

$$\delta = \tan^{-1}\left(\frac{\tan 2\epsilon}{\sin 2\tau}\right) = \tan^{-1}\left(\frac{\tan -90°}{\sin 2\tau}\right) = \tan^{-1}(-\infty) = -90°$$

And $\gamma = \frac{1}{2}\cos^{-1}(\cos 2\epsilon \cos 2\tau)$ (1-201)

$$= \frac{1}{2}\cos^{-1}(\cos(-90°)\cos 2\tau) = \frac{1}{2}\cos^{-1}(0) = \frac{90°}{2} = 45°$$

From (1-196) $\frac{E_2}{E_1} = \tan\gamma = \tan 45° = 1$

$\therefore$ $\underline{E_2 = E_1 \quad \delta = -90°}$

(c) Similar to the previous solution, for left circular $AR = +1$ so $\epsilon = +45°$ and $\delta = +90°$

And still $\gamma = 45°$ and $E_2/E_1 = 1$

$\therefore$ $\underline{E_2 = E_1 \quad \delta = +90°}$

(d) For elliptical with $E_1 = E_2$

$$\underline{\delta \neq \pm 90° \qquad \delta \neq 0° \text{ or } 180°}$$

(e) For elliptical with $\delta = 90°$

$$\underline{E_1 \neq E_2}$$

<u>1.9-2</u> From (1-199), the phasor E field is $\vec{E} = E_1\hat{x} + E_2 e^{j\delta}\hat{y}$

(a) Linear: $\vec{E} = E_1\hat{x} + E_2\hat{y}$

(b) RHCP: $\vec{E} = E_1\hat{x} + E_2 e^{j-90°}\hat{y} = E_1(\hat{x} - j\hat{y})$

1.9-2 (con't)

(c) LHCP: $\vec{E} = E_1\hat{x} + E_1 e^{j90^\circ}\hat{y} = E_1(\hat{x}+j\hat{y})$

(d) Elliptical with $E_1 = E_2$: $\vec{E} = E_1(\hat{x} + e^{j\delta}\hat{y})$

(e) Elliptical with $\delta = 90^\circ$: $\vec{E} = E_1\hat{x} + jE_2\hat{y}$

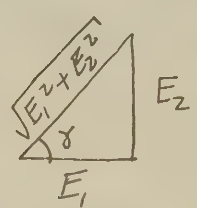
<u>1.9-3</u> Using (1-199)

$$|\vec{E}|^2 = \vec{E}\cdot\vec{E}^* = (E_1\hat{x} + E_2 e^{j\delta}\hat{y})\cdot(E_1\hat{x} + E_2 e^{-j\delta}\hat{y}) = E_1^2 + E_2^2$$

So $|\vec{E}| = \sqrt{E_1^2 + E_2^2}$

And $\vec{E} = E_1\hat{x} + E_2 e^{j\delta}\hat{y} = \sqrt{E_1^2+E_2^2}\left(\dfrac{E_1}{\sqrt{E_1^2+E_2^2}}\hat{x} + \dfrac{E_2}{\sqrt{E_1^2+E_2^2}} e^{j\delta}\hat{y}\right)$

Now



$$\cos\gamma = \frac{E_1}{\sqrt{E_1^2+E_2^2}}$$

$$\sin\gamma = \frac{E_2}{\sqrt{E_1^2+E_2^2}}$$

Thus $\vec{E} = \sqrt{E_1^2+E_2^2}\,(\cos\gamma\,\hat{x} + \sin\gamma\, e^{j\delta}\hat{y})$ which is (1-200).

<u>1.10-1</u>

From (1-214) $\lambda^2 = A_{em}\Omega_A$ or $\Omega_A = \lambda^2/A_{em}$

For an ideal dipole $A_{em} = 0.119\lambda^2$ (1-211)

So $\Omega_A = \lambda^2/0.119\lambda^2 = 1/0.119 = \boxed{8.40 \text{ steradians}}$

$$= 8.40\cdot\left(\frac{180 \text{ degree}}{\text{radian}}\right)^2 = \boxed{27{,}586.6 \text{ deg}^2}$$

<u>1.10-2</u>

From (1-213) $D = \dfrac{4\pi}{\lambda^2}A_{em}$ or $A_{em} = \dfrac{\lambda^2}{4\pi}D$

So $D_{dB} = 2.15 \Rightarrow D = 10^{2.15/10} = 1.64$ and

$$A_{em} = \frac{1.64}{4\pi}\lambda^2 = \boxed{0.131\lambda^2}$$

<u>1.10-3</u>

$\lambda = \dfrac{3\times10^8}{11.7\times10^9} = 2.564\times10^{-2}$ m $\qquad A_e = 6.30$ m²

From (1-222)

$$G = \frac{4\pi}{\lambda^2}A_e = \frac{4\pi}{(2.564\times10^{-2})^2}6.30 = 1.204\times10^5 = \boxed{50.8 \text{ dB}}$$

1.10-4 $d = 1.22\,m \qquad \lambda = 0.015\,m$ @ 20 GHz

The physical aperture area is

$$A_p = \frac{\pi}{4} d^2 = \frac{\pi}{4}(1.22)^2 = 1.169\,m^2$$

The effective aperture is

$$A_e = 0.55\,A_p = 0.55(1.169) = 0.643\,m^2$$

The gain is, from (1-222),

$$G = \frac{4\pi}{\lambda^2} A_e = \frac{4\pi}{(0.015)^2}\,0.643 = 35{,}911.9 = \boxed{45.55\,dB}$$

1.10-5

From (1-213) $\quad D = \frac{4\pi}{\lambda^2} A_{em} \quad$ or $\quad A_{em} = \frac{\lambda^2}{4\pi} D$

But

$$D = \frac{4\pi U_m}{P_r}$$

and

$$U_m = \frac{1}{2}\frac{|E|^2}{\eta} r^2 \qquad \text{and} \qquad P_r = \frac{1}{2}|I|^2 R_{ri}$$

So

$$A_{em} = \frac{\lambda^2}{4\pi}\,\frac{4\pi\,\frac{1}{2}\frac{|E|^2}{\eta} r^2}{\frac{1}{2}|I|^2 R_{ri}} = \frac{\lambda^2}{\eta}\,\frac{(90)^2 |I|^2 \frac{1}{r^2} r^2}{|I|^2 R_{ri}}$$

$$= \frac{\lambda^2}{120\pi}\,\frac{8100}{50} \qquad \text{where } R_{ri} = R_{in} = 50\,\Omega \text{ since lossless}$$

$$= \boxed{0.43\,\lambda^2}$$

1.10-6

$$P_R = P_T\,\frac{G_T G_R \lambda^2}{(4\pi r)^2} \qquad (1\text{-}223)$$

$$10\log_{10} P_R = 10\log P_T + 10\log G_T + 10\log G_R + 20\log\lambda - 20\log(4\pi) - 20\log r$$

$$P_R(dBW) = P_T(dBW) + G_T(dB) + G_R(dB) + 20\log\frac{3\times10^2\,m/s}{f(MHz)}$$
$$-20\log(4\pi) - 20\log(r(km)\times10^{-3})$$
$$= P_T(dBW) + G_T(dB) + G_R(dB) - 20\log f(MHz) - 20\log r(km)$$
$$+20\log 3\times10^2 - 20\log 4\pi + 20\log 10^{-3}$$
$$= P_T(dBW) + G_T(dB) + G_R(dB) - 20\log r(km)$$
$$-20\log f(MHz) - 32.44$$

or

$$P_R(dBm) = P_T(dBm) + G_T(dB) + G_R(dB) - 20\log r(km)$$
$$-20\log f(MHz) - 32.44 \qquad \text{which is (1-224).}$$

1.10-7 Working from (1-224)

$$P_R(dBm) = P_T(dBm) + G_T(dB) + G_R(dB) - 20\log f(MHz) \underbrace{-20\log\left[\frac{1.609\,km}{mi.}\, r(miles)\right] - 32.44}_{-20\log 1.609 - 20\log r(miles) - 32.44}$$

$$= P_T(dBm) + G_T(dB) + G_R(dB) - 20\log f(MHz) - 20\log r(miles) - 36.57$$

1.10-8

In Example 1-2 $P_T = 2\,W$, $G_T = 37\,dB = 5011.9$, $G_R = 45.8\,dB = 38{,}019.$

$\lambda = 0.015\,m$ @ $20\,GHz$ $\quad r = 36{,}941{,}031\,m$

Then (1-223) gives

$$P_R = P_T \frac{G_T G_R \lambda^2}{(4\pi r)^2} = 2\,\frac{(5011.9)(38{,}019)(0.015)^2}{(4\pi\, 3.6941031\times10^7)^2} = \boxed{3.98\times10^{-13}\,W}$$

1.10-9 $P_T(dBm) = 10\log 200 = 23$

$G_T = 19.3\,dB \qquad G_R = 50.4\,dB \qquad f = 11.7\,GHz \qquad r = 36{,}941\,km$

From (1-224)

$$P_R(dBm) = P_T(dBm) + G_T(dB) + G_R(dB) - 20\log r(km) - 20\log f(MHz) - 32.44$$

$$= 23 + 19.3 + 50.4 - 20\log(36{,}941) - 20\log(1.17\times10^4) - 32.44$$

$$= -112.45$$

$P_R = -142.45\,dBW \qquad P_R = 10^{\frac{-142.45}{10}} = \boxed{5.69\times10^{-15}\,W}$

1.10-10

$P_T(dBW) = 13 \qquad G_T(dB) = 10 \qquad G_R(dB) = 3 \qquad f = 150\,MHz$

$r = 50\,km$

From (1-224)

$$P_R(dBW) = P_T(dBW) + G_T(dB) + G_R(dB) - 20\log r(km) - 20\log f(MHz) - 32.44$$

$$= 13 + 10 + 3 - 20\log 50 - 20\log 150 - 32.44$$

$= -83.94\,dBW \qquad P_R = 10^{-8.394} = \boxed{4.036\times10^{-9}\,W}$

1.10-11

(a) 50 km of RG-8

$$L_{cable} = \frac{0.1\,dB}{m} \times 50\times10^3\,m = \boxed{5000\,dB}$$

1.10-11 (con't)

(b) Radio system of Prob. 1.10-10

Net gain = $P_R(dBW) - P_T(dBW) = -83.94 - 13 = -96.94\,dB$

Net loss = $\boxed{96.94\ dB}$

(c) Yes, repeaters would be necessary for the cable system.

(d) For a 500 m path

$$L_{cable} = \frac{0.1\,dB}{m} \times 500\,m = \boxed{50\,dB}$$

$$L_{radio} = -\left[G_T(dB) + G_R(dB) - 20\log r(km) - 20\log f(MHz) - 32.44\right]$$

$$= -\left[10 + 3 - 20\log 0.5 - 20\log 150 - 32.44\right] = \boxed{56.94\,dB}$$

(e) $f = 300\,MHz$, $r = 500\,m$, $\alpha = 0.14\,dB/m$

$$L_{cable} = 0.14\,\frac{dB}{m} \times 500\,m = \boxed{70\,dB}$$

$$L_{radio} = -\left[10 + 3 - 20\log 0.5 - 20\log 300 - 32.44\right] = \boxed{62.96\,dB}$$

(f) Fiber optic cable with $\alpha = 1\,dB/km$

$$L_{fiber} = 1\,\frac{dB}{km} \times 50\,km = \boxed{50\,dB}$$

$$L_{fiber} = 1\,\frac{dB}{km} \times 0.5\,km = \boxed{0.5\,dB}$$

(g)

	f = 150 MHz		300 MHz
	50 km	500 m	500 m
Cable (RG-8)	5000 dB	50 dB	70 dB
Radio	96.94 dB	56.94 dB	62.96 dB

1.10-12

$$P_R = P_T\,\frac{G_T G_R \lambda^2}{(4\pi r)^2} \qquad (1\text{-}223)$$

$$EIRP = P_T G_T \qquad (1\text{-}226)$$

$$G_R = \frac{4\pi}{\lambda^2} A_{eR} \qquad (1\text{-}222)$$

So

$$P_R = EIRP\,\frac{\frac{4\pi}{\lambda^2} A_{eR} \lambda^2}{4\pi 4\pi r^2} = \boxed{EIRP\,\frac{A_{eR}}{4\pi r^2}}$$

or

$$P_R(dBm) = EIRP(dBm) + 10\log A_{eR}(m^2) - 10\log(4\pi r^2)$$

1.10-13 Radar: $\sigma = 0.85\lambda^2$; $\lambda = 0.03\,m$ @ 10 GHz

$r = 1000\,m$; $P_T = 1000\,W$; $G_T(dB) = G_R(dB) = 20$, $G_T = G_R = 100$

From (1-233)

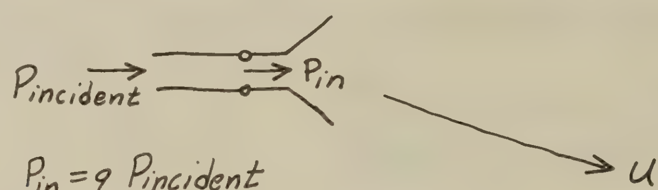
$$P_R = P_T \frac{\lambda^2 G_R G_T \sigma}{(4\pi)^3 r^4} = 10^3 \frac{(0.03)^2 (100)^2\, 0.85}{(4\pi)^3 (1000)^4} = \boxed{3.855 \times 10^{-12}\ W}$$

1.10-14 $P_T = 100\,kW$ $G_T(dB) = 2$ $G_T = 10^{2/10} = 1.585$

From (1-226)

$$EIRP = P_T G_T = (100\,kw)(1.585) = \boxed{158.5\,kW}$$

1.11-1



$P_{in} = q\, P_{incident}$

From (1-148) $$P_{in} = \frac{4\pi}{G(\theta,\phi)} U(\theta,\phi) = q\, P_{incident}$$

So $$U(\theta,\phi) = \frac{1}{4\pi} G(\theta,\phi)\, q\, P_{incident}$$

In a given direction (θ,ϕ)

$$\frac{U(VSWR)}{U(VSWR=1)} = \frac{q(VSWR)}{q(VSWR=1)}$$ is the radiation intensity relative to that when perfectly matched (VSWR=1).

$$= q(VSWR) \quad \text{since } q(VSWR=1) = 1$$

$$= 1 - \left| \frac{VSWR-1}{VSWR+1} \right|^2$$

Then

VSWR	$\frac{U(VSWR)}{U(VSWR=1)} = q$	Relative change in dB $10 \log q$
1	1	0 dB
1.01	0.9999752	−0.00011
1.2	0.991735	−0.03604
2.0	0.88888	−0.5115
10.0	0.33058	−4.8072

1.11-2

For a 2dB axial ratio $|AR| = 10^{2/20} = 10^{0.1} = 1.259$
Since the sense is right hand, sign (AR) = −1. So

$$AR = -1.259$$

From (1-197)

$$\epsilon = \cot^{-1}(AR) = \cot^{-1}(-1.259) = -38.46°$$

From (1-201) and (1-202)

$$\gamma = \tfrac{1}{2}\cos^{-1}(\cos 2\epsilon \cos 2\tau) = \tfrac{1}{2}\cos^{-1}[\cos(2(-38.46°))\cos(90°)] = 45°$$

$$\delta = \tan^{-1}\left(\frac{\tan 2\epsilon}{\sin 2\tau}\right) = \tan^{-1}\left[\frac{\tan(2(-38.46°))}{\sin(90°)}\right] = -76.92°$$

Now (1-249) can be used to obtain the complex unit vector.

$$\hat{e} = \cos\gamma\,\hat{x} + \sin\gamma\, e^{j\delta}\hat{y} = \cos(45°)\hat{x} + \sin(45°)e^{j\,-76.92°}\hat{y}$$

$$= \underline{0.707(\hat{x} + e^{-j76.92°}\hat{y})}$$

1.11-3

(a) Horizontal linear

$$\hat{h}_R = \hat{x}$$

$$p = |\hat{h}_R^* \cdot \hat{e}|^2 = |\hat{x}\cdot 0.707(\hat{x} + e^{-j76.92°}\hat{y})|^2 = |0.707|^2 = \boxed{0.5}$$

(b) Vertical linear

$$\hat{h}_R = \hat{y}$$

$$p = |\hat{y}\cdot 0.707(\hat{x} + e^{-j76.92°}\hat{y})|^2 = |0.707\, e^{-j76.92°}|^2 = \boxed{0.5}$$

(c) Right hand circular

$\hat{h}_R = 0.707(\hat{x} - j\hat{y})$ see Prob. 1.9-2

$$P = |0.707(x \underset{=}{+} j\hat{y})\cdot 0.707(\hat{x} + e^{-j76.92°}\hat{y})|^2$$

$$= 0.25\,|1 + j e^{-j76.92°}|^2 = 0.25[(1+\sin 76.92°)^2 + \cos^2 76.92°]$$

$$= 0.25[3.9481] = \boxed{0.987}$$

(d) Left hand circular

$$\hat{h}_R = 0.707(\hat{x} + j\hat{y})$$

$$p = |0.707(\hat{x} \underset{=}{-} j\hat{y})\cdot 0.707(\hat{x} + e^{-j76.92°}\hat{y})|^2$$

$$= 0.25[(1 - \sin 76.92°)^2 + \cos^2 76.92°] = \boxed{0.01297}$$

1.11-3 (con't)

(e) Right hand elliptical with $AR(dB) = 2$ and $\tau = 45^\circ$

$$\hat{h}_R = 0.707(\hat{x} + e^{-j76.92^\circ}\hat{y}) \qquad \text{see Prob. 1.11-2}$$

$$p = |0.707(\hat{x} + e^{\pm j76.92^\circ}\hat{y}) \cdot 0.707(\hat{x} + e^{-j76.92^\circ}\hat{y})|^2$$

$$= 0.25|1+1|^2 = \boxed{1}$$

(f) Left elliptical with $AR(dB) = 2$ and $\tau = 135^\circ$.

In a fashion similar to that used to solve Prob. 1.11-2

$$AR = \pm 1.259 \quad \text{and} \quad \epsilon = \pm 38.46^\circ$$

Then

$$\gamma = \tfrac{1}{2}\cos^{-1}[\cos(2(38.46^\circ))\underbrace{\cos(2(135^\circ))}_{0}] = 45^\circ$$

$$\delta = \tan^{-1}\left[\frac{\tan 2(38.46^\circ)}{\sin 2(135^\circ)}\right] = \tan^{-1}\left[\frac{\tan 76.92^\circ}{-1}\right] = 103.08^\circ$$

Thus

$$\hat{h}_R = 0.707(\hat{x} + e^{j103.08^\circ}\hat{y})$$

And

$$p = |0.707(\hat{x} + e^{-j103.08^\circ}\hat{y}) \cdot 0.707(\hat{x} + e^{-j76.92^\circ}\hat{y})|^2$$

$$= 0.25|1 + e^{j180^\circ}|^2 = 0.25|1-1|^2 = \boxed{0}$$

1.11-4

Taking 10 log of both sides of

$$E_{rms}^2 = f^2 \frac{1}{G} V_{in,rms}^2 \frac{\eta 4\pi}{R_r c^2}$$

gives

$$20 \log E_{rms} = 20 \log f(\text{Hz}) - 10 \log G + 20 \log V_{in,rms} - 10 \log R_r + 10 \log \frac{\eta 4\pi}{c^2}$$

$$= 20 \log[f(\text{MHz}) \times 10^6] - G(dB) + 20 \log V_{in,rms} - 10 \log R_r - 10 \log(377 \cdot 4\pi/(3\times10^8)^2)$$

$$= 20 \log f(\text{MHz}) - G(dB) + 20 \log V_{in,rms} - 10 \log R_r - 12.8$$

which is (1-261)

1.11-5

Receiving antenna: linearly polarized, $Z_{in} = 300 + j0\,\Omega$, $R_r = 300\,\Omega$, $G = 1.64$

T.L.: $Z_0 = 300\,\Omega$ Receiver input: $200\,\mu V$ (peak)

(a) $R_{in} = R_r + R_{ohmic}$ or $300 = 300 + R_{ohmic} \Rightarrow R_{ohmic} = 0$

1.11-5 (con't)

Then $e = \frac{R_r}{R_{in}} = \frac{R_r}{R_r + R_{ohmic}} = \frac{300}{300} = \boxed{1}$

And $\boxed{q = 1}$ since the antenna and T.L. are matched.

(b)

From (1-254) $p = 0.5$ for a circularly polarized transmitter.

$V_{peak}/Z \approx 300 \approx 46$

Modifying (1-263) for peak values and including polarization loss as in (1-255) gives $P_r = 66\,pW$ $P_r = P_{density} A_{eff}$

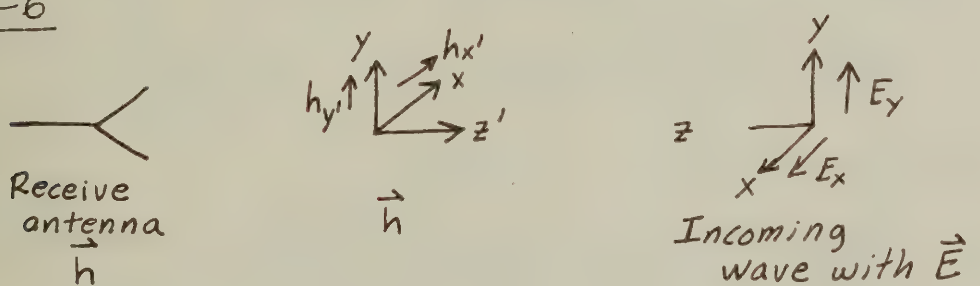
$$E_{peak}(dB\mu V/m) = 20\log f(MHz) - G(dB) + V_{in,peak}(dB\mu V) - 10\log p - 37.6$$ $A_{eff} = \frac{G\lambda^2}{4\pi}$

$$\underbrace{\quad}_{\text{minimum field}} = 20\log(100) - 10\log(1.64) + 20\log(\underbrace{200}_{\text{minimum voltage}}) - 10\log(0.5) - 37.6$$ for circular polarization

$$= 40 - 2.15 + 46.02 + 3.01 - 37.6 = 49.28 \text{ dB}\mu V/m$$ $P_r = 66 \times 2 = 132\,fW/m^2$ required

$$E_{peak}(\text{minimum}) = 10^{49.28/20} = \boxed{291.1\ \mu V/m}$$ OK

1.11-6



For the antenna

$$\vec{h} = |h_{x'}|\hat{x}' + |h_{y'}|e^{j\delta}\hat{y}'$$

and in the xy-system

$$\vec{h} = |h_x|\hat{x} + |h_y|e^{-j\delta}\hat{y}$$ since y-component now lags x-component by δ since x-axis is reversed.

$= \vec{h}^*$ relative to xy-system.

CHAPTER 2

2.1-1

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{e^{-j\beta(r-\frac{\Delta z}{2}\cos\theta)}}{r-\frac{\Delta z}{2}\cos\theta} - \frac{e^{-j\beta(r+\frac{\Delta z}{2}\cos\theta)}}{r+\frac{\Delta z}{2}\cos\theta}\right]$$

$$\approx \frac{q}{4\pi\epsilon_0} e^{-j\beta r}\,\frac{(r+\frac{\Delta z}{2}\cos\theta)e^{j\frac{\beta\Delta z}{2}\cos\theta} - (r-\frac{\Delta z}{2}\cos\theta)e^{-j\frac{\beta\Delta z}{2}\cos\theta}}{r^2}$$

where $r^2-(\frac{\Delta z}{2}\cos\theta)^2 \approx r^2$ was used since $r >> \Delta z$

$$= \frac{q}{4\pi\epsilon_0}\,\frac{e^{-j\beta r}}{r^2}\left[r\,2j\sin\left(\frac{\beta\Delta z}{2}\cos\theta\right) + \frac{\Delta z}{2}\cos\theta\,2\cos\left(\frac{\beta\Delta z}{2}\cos\theta\right)\right]$$

$$\approx \frac{q}{4\pi\epsilon_0}\,\frac{e^{-j\beta r}}{r^2}\left[2jr\,\frac{\beta\Delta z}{2}\cos\theta + \frac{\Delta z}{2}\cos\theta\,2\right]$$

using $\beta\Delta z = \frac{2\pi\Delta z}{\lambda} << 1$ since $\lambda >> \Delta z$.

$$= \frac{q}{4\pi\epsilon_0}\,\frac{e^{-j\beta r}}{r^2}(j\beta r+1)\,\Delta z\cos\theta$$

But $I = j\omega q$ so $q = I/j\omega$

Then $$\Phi = \frac{e^{-j\beta r}}{4\pi r^2}\,\frac{I\Delta z}{j\omega\epsilon_0}(1+j\beta r)\cos\theta$$

Now $$\vec{E} = -j\omega\mu\vec{A} - \nabla\Phi \qquad (1\text{-}39)$$

And $$\vec{A} = \frac{I\Delta z\, e^{-j\beta r}}{4\pi r}\hat{z} \qquad (1\text{-}60)$$

So

$$E_r = -j\omega\mu A_r - \frac{\partial\Phi}{\partial r} = -j\omega\mu\frac{I\Delta z e^{-j\beta r}}{4\pi r}\cos\theta - \frac{I\Delta z}{j\omega\epsilon_0}\frac{1}{4\pi}\frac{\partial}{\partial r}\left(\frac{e^{-j\beta r}}{r^2}+\frac{j\beta e^{-j\beta r}}{r}\right)\cos\theta$$

$$= \frac{I\Delta z}{4\pi j\omega\epsilon_0}\cos\theta\left[\omega^2\mu\epsilon_0\frac{e^{-j\beta r}}{r} - \left(\frac{-j\beta e^{-j\beta r}}{r^2} - 2\frac{e^{-j\beta r}}{r^3} + \frac{\beta^2 e^{-j\beta r}}{r} - \frac{j\beta e^{-j\beta r}}{r^2}\right)\right]$$

$$= \frac{I\Delta z\, e^{-j\beta r}}{4\pi r j\omega\epsilon_0}\cos\theta\left[\beta^2+\frac{j\beta}{r}+\frac{2}{r^2}-\beta^2+\frac{j\beta}{r}\right] = \frac{I\Delta z}{2\pi}\,\frac{e^{-j\beta r}}{r}\,\frac{\cos\theta}{j\omega\epsilon_0}\left(\frac{j\beta}{r}+\frac{1}{r^2}\right)$$

$$= \frac{I\Delta z}{2\pi}\,\frac{e^{-j\beta r}}{r}\left(\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\right)\cos\theta \quad \text{as in (1-70).}$$

And

$$E_\theta = -j\omega\mu A_\theta - \frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -j\omega\mu(-A_z\sin\theta) - \frac{1}{r}\frac{q}{4\pi\epsilon_0}\frac{e^{-j\beta r}}{r^2}(j\beta r+1)\Delta z(-\sin\theta)$$

$$= \left[j\omega\mu\frac{I\Delta z}{4\pi r} + \frac{I\Delta z}{j\omega 4\pi\epsilon_0}\frac{1}{r^3}(j\beta r+1)\right]e^{-j\beta r}\sin\theta$$

2.1-1 (con't)

$$E_\theta = \frac{I\Delta z}{4\pi} j\omega\mu\left[1 + \frac{j\beta}{j\omega\epsilon_0 j\omega\mu}\frac{1}{r} + \frac{1}{j\omega\epsilon_0 j\omega\mu}\frac{1}{r^2}\right]\frac{e^{-j\beta r}}{r}\sin\theta$$

$$= \frac{I\Delta z}{4\pi} j\omega\mu\left[1 + \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2}\right]\frac{e^{-j\beta r}}{r}\sin\theta \quad \text{as in (1-70).}$$

<u>2.1-2</u> $\vec{J} = \hat{z} J_o \sin\left[\beta\left(\frac{\Delta z}{2} - |z|\right)\right]$

From (1-20)

$$\nabla\cdot\vec{J} = -j\omega\rho$$

Or

$$\rho = \frac{1}{-j\omega}\nabla\cdot\vec{J} = \frac{j}{\omega}\frac{\partial J_z}{\partial z} = \frac{j}{\omega} J_o \frac{\partial}{\partial z}\sin\left[\beta\left(\frac{\Delta z}{2} - |z|\right)\right]$$

$$= \frac{j}{\omega} J_o \cos\left[\beta\left(\frac{\Delta z}{2} - |z|\right)\right]\frac{\partial}{\partial z}(-\beta|z|) = \frac{j}{\omega} J_o \cos\left[\beta\left(\frac{\Delta z}{2} - |z|\right)\right]\frac{-\beta z}{|z|}$$

$$= \frac{-j\beta}{\omega} J_o \frac{z}{|z|}\cos\left[\beta\left(\frac{\Delta z}{2} - |z|\right)\right]$$

<u>2.1-3</u>

For a parallel plate capacitor

$$C = \frac{\epsilon A}{d} \qquad A = \text{area of plates} \qquad d = \text{plate separation}$$

In this case

$$C = \frac{\epsilon_0 \pi (\Delta r)^2}{\Delta z}$$

<u>2.1-4</u>

(a) $\Delta r = 0.01\lambda \qquad \Delta z = 0.02\lambda$

$$C = \frac{\epsilon_0 \pi (\Delta r)^2}{\Delta z} = \frac{8.85\times10^{-12}\,\pi(0.01\lambda)^2}{0.02\lambda} = 1.390\times10^{-13}\lambda$$

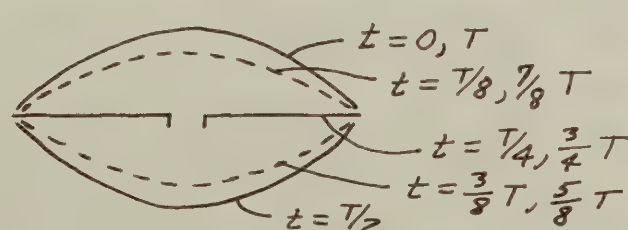
$$X_c = \frac{1}{\omega C} = \frac{\lambda}{2\pi c C} = \frac{\lambda}{2\pi\, 3\times10^8\, 1.39\times10^{-13}\lambda} = \boxed{3{,}817 \text{ ohms}}$$

(b) From (1-180)

$$R_r = 80\pi^2\left(\frac{\Delta z}{\lambda}\right)^2 = 80\pi^2(0.02)^2 = \boxed{0.316 \text{ ohm}}$$

<u>2.2-1</u>

Current on a half-wave dipole.



2.2-2

From (2-9) $f(\theta) = \dfrac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$

At $\theta = \frac{\pi}{2}$

$$f(\theta=\tfrac{\pi}{2}) = \frac{\cos(\frac{\pi}{2}\cos\frac{\pi}{2})}{\sin^2\frac{\pi}{2}} = \frac{\cos 0}{(1)^2} = 1 \qquad \text{Q.E.D.}$$

2.2-3

See Fig. 2-5b.

2.2-4

(a)

$$P_{ohmic} = \frac{R_s}{2\pi a}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1}{2}|I(z)|^2 dz \qquad (1\text{-}190)$$

For the half-wave dipole

$$I(z) = I_m \sin[\beta(\tfrac{\lambda}{4} - |z|)] \qquad (2\text{-}3)$$

So

$$\frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} |I(z)|^2 dz = \frac{1}{2} I_m^2 \, 2\int_0^{\lambda/4} \sin^2[\beta(\tfrac{\lambda}{4} - z)]\, dz$$

$$= I_m^2 \int_0^{\lambda/4} \frac{1}{2}[1 - \cos(2\beta(\tfrac{\lambda}{4} - z))]dz$$

$$= I_m^2 \frac{1}{2}\left[z - \frac{\sin(2\beta(\frac{\lambda}{4}-z))}{-2\beta\frac{\lambda}{4}}\right]_0^{\lambda/4} = I_m^2 \frac{1}{2}\left[\frac{\lambda}{4} + \frac{0 - \sin\pi}{2\beta\lambda/4}\right] = \frac{1}{2}\frac{\lambda}{4} I_m^2$$

Thus

$$P_{ohmic} = \frac{R_s}{2\pi a} \cdot \frac{1}{2} I_m^2 \frac{\lambda}{4}$$

$$R_{ohmic} = \frac{P_{ohmic}}{\frac{1}{2}|I_{in}|^2} = \frac{P_{ohmic}}{\frac{1}{2}I_m^2} = \boxed{\frac{R_s}{2\pi a}\frac{\lambda}{4}}$$

(b) which is half the value it would be for uniform current because

$$R_{ohmic}(\text{uniform}) \approx \frac{R_s}{2\pi a} L = \frac{R_s}{2\pi a}\frac{\lambda}{2} \qquad \text{from } (1\text{-}184)$$

2.2-5

$f = 10^8$ Hz, $\mu = \mu_0 = 4\pi \times 10^{-7} \frac{h}{m}$, $\sigma = 3.5 \times 10^7$ mho/m, $2a = 6.35 \times 10^{-3}$ m

From (1-185)

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}} = \sqrt{\frac{2\pi 10^8 \; 4\pi\times 10^{-7}}{2\,(3.5\times 10^7)}} = 3.3585 \times 10^{-3} \text{ ohm}$$

From Prob. 2.2-4a

$$R_{ohmic} = \frac{R_s}{2\pi a}\frac{\lambda}{4} = \frac{3.3585\times 10^{-3}}{2\pi \frac{6.35\times 10^{-3}}{2}}\frac{3}{4} = 0.126 \text{ ohm}$$

Thus

$$e \doteq \frac{R_r}{R_r + R_{ohmic}} = \frac{70}{70 + 0.126} = \boxed{99.82\,\%}$$

2.3-1

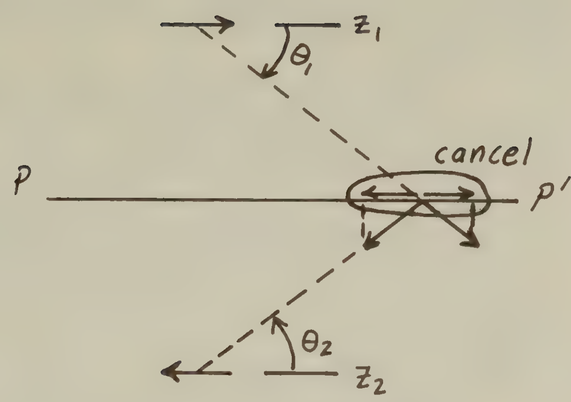
r-components of $\vec{E}$:

$E_{r1} = C\cos\theta_1$

$E_{r2} = -C\cos\theta_2 = -C\cos\theta_1$

where the minus sign arises because of the oppositely directed dipole.

We see that the tangential components of E_{r1} and E_{r2} cancel.

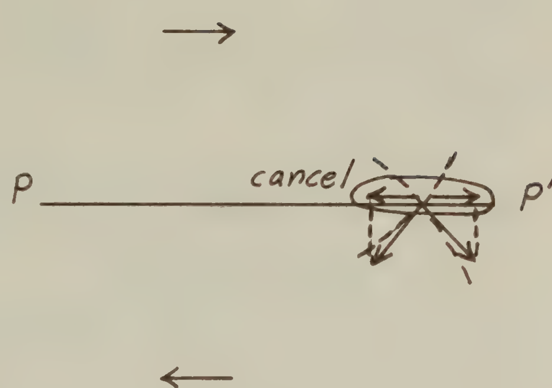


θ-components of $\vec{E}$:

$E_{\theta 1} = D\sin\theta_1$

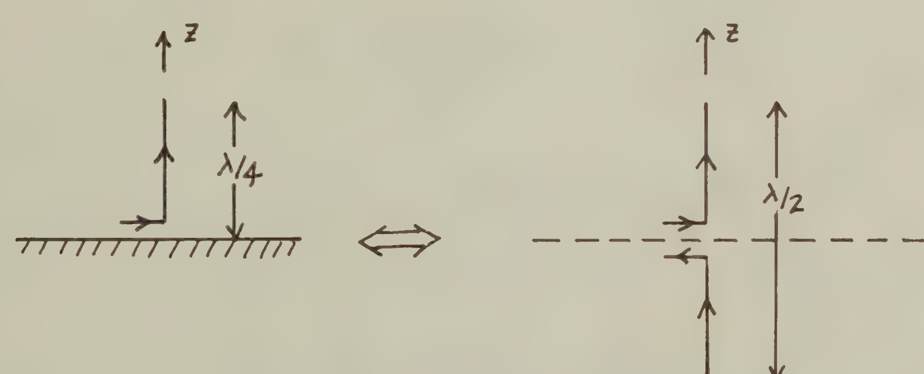
$E_{\theta 2} = -D\sin\theta_2 = -D\sin\theta_1$

and the projections of $E_{\theta 1}$ and $E_{\theta 2}$ onto the plane PP' cancel.

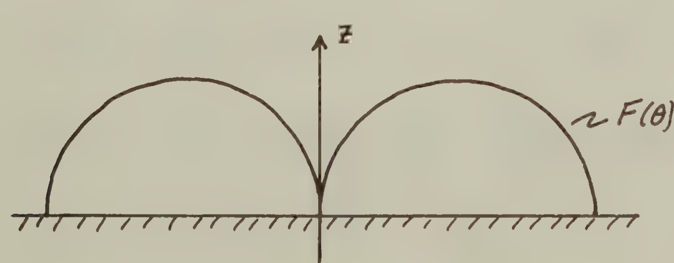


2.3-2

(a)



This equivalence (above the ground plane) shows that the patterns will be the same above the ground plane:



2.3-2 (con't)

(b) From (2-22) $D_{mono} = 2\, D_{dipole}$

So

$$D_{\frac{\lambda}{4}\,mono} = 2\, D_{\frac{\lambda}{2}\,dipole} = 2(1.64) = \boxed{3.28}$$

(c) From (2-19) $Z_{in,\,mono} = \frac{1}{2} Z_{in,\,dipole}$

So

$$Z_{in,\frac{\lambda}{4}\,mono} = \frac{1}{2} Z_{in,\frac{\lambda}{2}\,dipole} = \frac{1}{2}(73 + j\,42.5) = \boxed{36.5 + j21.25 \text{ ohms}}$$

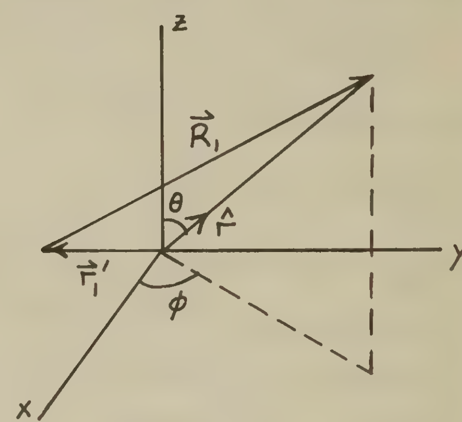
2.4-1

Eqn. (1-93) is

$$R = r - \hat{r}\cdot\vec{r}\,'$$

Also from (A-4)

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$



For R_1

$$\vec{r}_1\,' = -\frac{l}{2}\hat{y}$$

$$R_1 = r - \hat{r}\cdot\left(-\frac{l}{2}\hat{y}\right)$$

$$= r + \frac{l}{2}\sin\theta\sin\phi$$

For R_3

$$\vec{r}_3\,' = \frac{l}{2}\hat{y}$$

$$R_3 = r - \hat{r}\cdot\left(\frac{l}{2}\hat{y}\right) = r - \frac{l}{2}\sin\theta\sin\phi$$

For R_2

$$\vec{r}_2\,' = \frac{l}{2}\hat{x}$$

$$R_2 = r - \hat{r}\cdot\left(\frac{l}{2}\hat{x}\right) = r - \frac{l}{2}\sin\theta\cos\phi$$

For R_4

$$\vec{r}_4\,' = -\frac{l}{2}\hat{x}$$

$$R_4 = r - \hat{r}\cdot\left(-\frac{l}{2}\hat{x}\right) = r + \frac{l}{2}\sin\theta\cos\phi$$

2.4-2

Substituting (2-53) and (2-54)

$$E_\phi = \eta\beta^2 S\,\frac{I e^{-j\beta r}}{4\pi r}\sin\theta \qquad H_\theta = -\beta^2 S\,\frac{I e^{-j\beta r}}{4\pi r}\sin\theta$$

into

$$P_r = \frac{1}{2}\,\text{Re}\iint\left[E_\theta H_\phi^{*} - E_\phi H_\theta^{*}\right] r^2\sin\theta\, d\theta\, d\phi \qquad (1\text{-}125)$$

yields

2.4-2 (con't)

$$P_r = \frac{1}{2}\eta\beta^4 S^2 I^2 \frac{1}{(4\pi)^2}(-1)\,Re\iint\left(-\frac{e^{-j\beta r}}{r}\frac{e^{+j\beta r}}{r}\sin^2\theta\right)r^2\sin\theta\, d\theta\, d\phi$$

$$= \frac{\eta\beta^4 S^2 I^2}{32\pi^2}\underbrace{\int_0^{\pi}\sin^3\theta\, d\theta}_{4/3}\underbrace{\int_0^{2\pi}d\phi}_{2\pi} = \frac{\eta\beta^4 S^2 I^2}{4\pi\cdot 3} = \frac{120\pi}{12\pi}(\beta^2 S I)^2$$

$= 10(\beta^2 S I)^2$ which is (2-56).

2.4-3

$$R_{ohmic} = \frac{2\ell_1\ell_2}{\pi d^2}R_s\left\{\frac{1}{\sqrt{\left(\frac{\ell_1}{d}\right)^2-1}} + \frac{1}{\sqrt{\left(\frac{\ell_2}{d}\right)^2-1}}\right\} \qquad (2\text{-}60)$$

$$\approx \frac{2\ell_1\ell_1}{\pi d^2}R_s\left\{\frac{1}{\frac{\ell_1}{d}} + \frac{1}{\frac{\ell_2}{d}}\right\} \quad \text{if } \ell_1 \gg d \text{ and } \ell_2 \gg d$$

$$= \frac{2\ell_1\ell_2}{\pi d}R_s\,\frac{\ell_1+\ell_2}{\ell_1\ell_2} = \frac{2(\ell_1+\ell_2)}{\pi d}R_s \quad \text{which is (2-61).}$$

2.4-4

$\lambda = 300$ m @ 1 MHz $\quad b = 0.2$ m $\quad d = 8.12\times10^{-4}$ m

$S = \pi b^2 = \pi(0.2)^2 = 0.12566\ m^2$

From (2-57)

$$R_r = 31{,}200\left(\frac{S}{\lambda^2}\right)^2 = 31{,}200\left(\frac{0.12566}{300^2}\right)^2 = 6.082\times10^{-8} \text{ ohm}$$

From (2-63)

$$R_{ohmic} = \frac{2b}{d}R_s = \frac{2(0.2)}{8.12\times10^{-4}}\,2.632\times10^{-4} = 0.1297 \text{ ohm}$$

since $R_s = \sqrt{\frac{\omega\mu}{2\sigma}} = \sqrt{\frac{2\pi\times10^6\cdot 4\pi\times10^{-7}}{2\cdot 5.7\times10^7}} = 2.632\times10^{-4}$ ohm

Thus

$$e = \frac{R_r}{R_r + R_{ohmic}} = \frac{6.082\times10^{-8}}{6.082\times10^{-8}+0.1297} = 4.69\times10^{-7} = \boxed{4.69\times10^{-5}\,\%}$$

2.4-5

$b = 0.2$ m $\quad d = 8.12\times10^{-4}$ m

From (2-65)

$$L = b\mu\left[\ln\left(\frac{16b}{d}\right) - 1.75\right] = (0.2)(4\pi\times10^{-7})\left[\ln\left(\frac{3.2}{8.12\times10^{-4}}\right) - 1.75\right]$$

$$= 1.64\times10^{-6} = \boxed{1.64\ \mu H}$$

2.4-6

$\lambda = 300\,m$ @ 1 MHz $\quad b = 0.15\,m \quad d = 0.003\,m$

From (2-57)

$$R_r = 31{,}200\left(\frac{\pi b^2}{\lambda^2}\right)^2 = 31{,}200\left[\frac{\pi(0.15)^2}{300^2}\right]^2 = \underline{1.9246\times10^{-8}\ \text{ohm}}$$

From (2-65)

$$X_{in} = \omega L = \omega b \mu\left[\ln\left(\frac{16b}{d}\right) - 1.75\right]$$

$$= 2\pi\times10^6\,(0.15)\,4\pi\times10^{-7}\left[\ln\left(\frac{16(0.15)}{0.003}\right) - 1.75\right] = \underline{5.844\ \text{ohm}}$$

And

$$R_s = \sqrt{\frac{\pi f \mu_0}{\sigma}} = \sqrt{\frac{\pi\,10^6\,4\pi\times10^{-7}}{5.7\times10^7}} = 2.6317\times10^{-4}\ \text{ohm}$$

So from (2-63)

$$R_{ohmic} = \frac{2b}{d}R_s = \frac{2(0.15)}{0.003}\,2.6317\times10^{-4} = \underline{0.02632\ \text{ohm}}$$

Thus

$$Z_{in} = \underbrace{R_r + R_{ohmic}}_{R_{in}} + jX_{in} = \boxed{0.02632 + j5.844\ \text{ohms}}$$

and

$$e = \frac{R_r}{R_{in}} = \frac{1.9246\times10^{-8}}{0.02632} = \boxed{7.3\times10^{-5}\ \%}$$

2.4-7

$\lambda = 300\,m$ @ 1 MHz $\quad L = 20\,cm \quad b = 1\,cm \quad n = 22 \quad \mu_{eff} = 38$

From (2-59)

$$R_r = 31{,}200\left[n\mu_{eff}\frac{S}{\lambda^2}\right]^2 = 31{,}200\left[22\cdot38\cdot\frac{\pi(0.01)^2}{300^2\,4}\right]^2 = \boxed{1.66\times10^{-8}\ \text{ohm}}$$

2.4-8

$\lambda = 10\,m$ @ 30 MHz, $S = (1\,m)^2 = 1\,m^2 = \frac{1\,m^2}{(10\,m)^2}\lambda^2 = 0.01\lambda^2$, $d = 0.02\,m$

(a) From (2-57)

$$R_r = 31{,}200\left(\frac{S}{\lambda^2}\right)^2 = 31{,}200\left(\frac{0.01\lambda^2}{\lambda^2}\right)^2 = \boxed{3.12\ \text{ohms}}$$

(b) From (2-64)

$$L = \frac{\mu}{\pi}\left[\ell_2\cosh^{-1}\frac{\ell_1}{d} + \ell_1\cosh^{-1}\frac{\ell_2}{d}\right] = \frac{4\pi\times10^{-7}}{\pi}\left[2\ell\cosh^{-1}\frac{\ell}{d}\right]$$

$$= 8\times10^{-7}(1)\cosh^{-1}\frac{1}{0.02} = 3.68\times10^{-6}$$

$$X_{in} = \omega L = 2\pi\,3\times10^7\,3.68\times10^{-6} = \boxed{694\ \text{ohms}}$$

(c) $$R_s = \sqrt{\frac{\pi f\mu_0}{\sigma}} = \sqrt{\frac{\pi\,3\times10^7\,4\pi\times10^{-7}}{3.5\times10^7}} = \overset{1.8}{\cancel{1.06}}\times10^{-3}$$

$$R_{ohmic} = \frac{2(\ell_1+\ell_2)}{\pi d}R_s = \frac{2(2)}{\pi(0.02)}\,1.06\times10^{-3} = \boxed{\underset{0.114}{\cancel{6.76}}\times10^{-2}\ \text{ohm}}$$

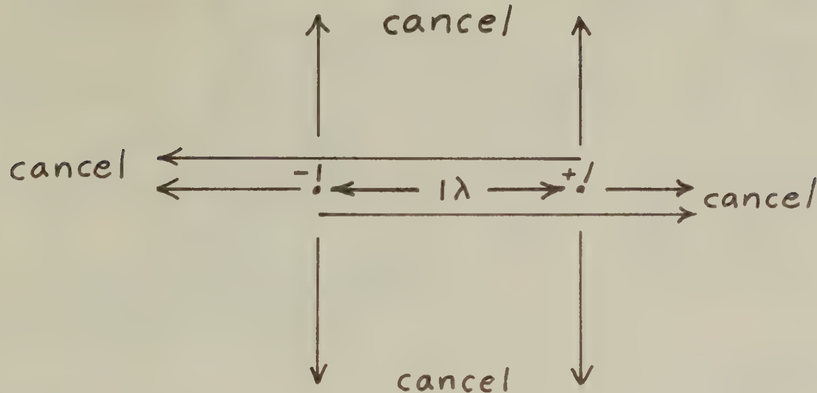
$$e = \frac{R_r}{R_{in}} = \frac{R_r}{R_r + R_{ohmic}}$$

$$e = \frac{3.12}{3.188} = \boxed{97.88\%}$$

CHAPTER 3

3.1-1

(a)

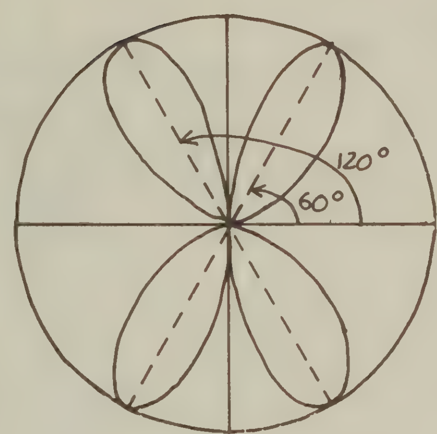


The waves add (perfectly) when the path difference is $\pm\lambda/2$, which compensates for the $180°$ current phase difference:

$\lambda \cos\theta_m = \pm \frac{\lambda}{2}$

$\cos\theta_m = \pm \frac{1}{2}$

$\theta_m = 60°, 120°$



(b)

$$AF = -1 e^{-j\beta\frac{d}{2}\cos\theta} + 1 e^{j\beta\frac{d}{2}\cos\theta}$$

$$= 2j\sin\left(\beta\frac{d}{2}\cos\theta\right) = \boxed{2j\sin(\pi\cos\theta)} \quad \text{since } d=\lambda$$

(c)

$$1 = |\sin(\pi\cos\theta_m)| \Rightarrow \pm\frac{\pi}{2} = \pi\cos\theta_m \text{ or } \pm\frac{1}{2} = \cos\theta_m$$

$$\therefore \boxed{\theta_m = 60°, 120°}$$

(d)

$$|f(\theta)| = \frac{|AF(\theta)|}{|AF(\theta_m)|} = \frac{|2j\sin(\pi\cos\theta)|}{|2j\sin(\pi\cos\theta_m)|} = \boxed{|\sin(\pi\cos\theta)|}$$

(e)

$$\psi = \beta d\cos\theta + \alpha = 2\pi\cos\theta + \pi \qquad \text{since } d=\lambda \text{ \& } \alpha = 180°$$

Then (3-20) is

$$f(\psi) = \cos\left(\frac{\psi}{2}\right) = \cos\left(\pi\cos\theta + \frac{\pi}{2}\right) = -\sin(\pi\cos\theta)$$

$\therefore |f(\theta)| = |\sin(\pi\cos\theta)|$ Note: in (d) $\overset{\bullet}{180°}$ $\overset{\bullet}{0°}$ and in (e) $\overset{\bullet}{0°}$ $\overset{\bullet}{180°}$

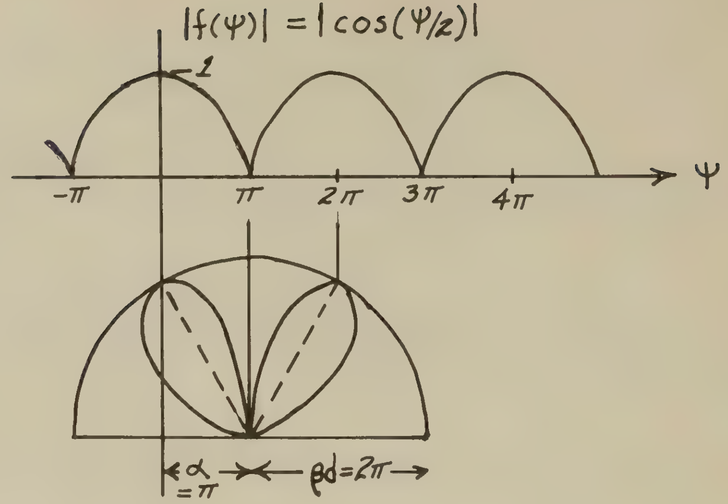
3.1-2

$d = \lambda$

$\beta d = \frac{2\pi}{\lambda}\lambda = 2\pi$

$\alpha = \pi$

(Note: also could use $\alpha = -\pi$.)

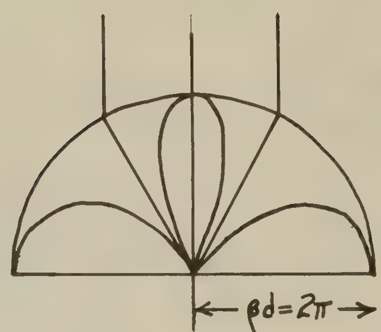


3.1-3

$d = \lambda$

$\beta d = \frac{2\pi}{\lambda}\lambda = 2\pi$

$\alpha = 0$



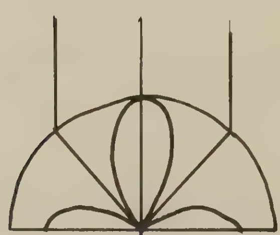
3.1-4

(a)

$d = \frac{3}{4}\lambda$

$\beta d = \frac{3}{2}\pi$

$\alpha = 0$

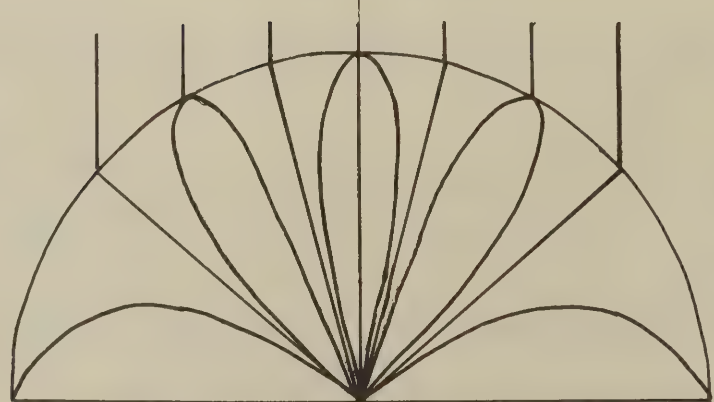


(b)

$d = 2\lambda$

$\beta d = 4\pi$

$\alpha = 0$



3.1-5

$F=\cos\frac{\psi}{2}$ for a broadside array of two isotropic elements with $\psi=\beta d\cos\theta$

$$\Omega_A=\int_0^{2\pi}\int_0^{\pi}|F(\theta)|^2\sin\theta\,d\theta\,d\phi=\int_0^{2\pi}d\phi\int_{\beta d}^{-\beta d}\cos^2\frac{\psi}{2}\,\frac{-d\psi}{\beta d}\qquad \text{since } d\psi=-\beta d\sin\theta$$

$$=\frac{1}{\beta d}(2\pi)\int_{-\beta d}^{\beta d}\tfrac{1}{2}(1+\cos\psi)\,d\psi=\frac{\pi}{\beta d}\left[\psi+\sin\psi\right]_{-\beta d}^{\beta d}$$

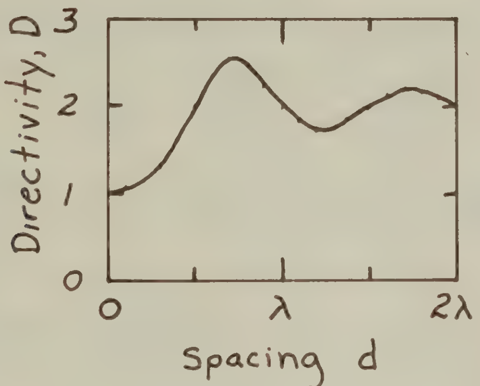
$$=\frac{\pi}{\beta d}\left[2\beta d+2\sin\beta d\right]=2\pi\left(1+\frac{\sin\beta d}{\beta d}\right)$$

$$D=\frac{4\pi}{\Omega_A}=\boxed{\frac{2}{1+\frac{\sin\beta d}{\beta d}}}$$

3.1-6

Plot of

$$D=\frac{2}{1+\frac{\sin\beta d}{\beta d}}$$



3.2-1

Using (3-33)

$$f(\pi-\Delta)=\frac{\sin\left(N\frac{\pi}{2}-N\frac{\Delta}{2}\right)}{N\sin\left(\frac{\pi}{2}-\frac{\Delta}{2}\right)}\qquad \text{since } f(\psi)=\frac{\sin\left(\frac{N}{2}\psi\right)}{N\sin\left(\frac{1}{2}\psi\right)}$$

$$=\frac{\sin N\frac{\pi}{2}\cos\left(N\frac{\Delta}{2}\right)-\cos N\frac{\pi}{2}\sin N\frac{\Delta}{2}}{N\cos\frac{\Delta}{2}}$$

$$=\begin{cases}-\cos N\frac{\pi}{2}\sin N\frac{\Delta}{2}\Big/N\cos\frac{\Delta}{2} & N \text{ even}\\ \sin N\frac{\pi}{2}\cos N\frac{\Delta}{2}\Big/N\cos\frac{\Delta}{2} & N \text{ odd}\end{cases}$$

Similarly

$$f(\pi+\Delta)=\begin{cases}+\cos N\frac{\pi}{2}\sin N\frac{\Delta}{2}\Big/N\cos\frac{\Delta}{2} & N \text{ even}\\ \sin N\frac{\pi}{2}\cos N\frac{\Delta}{2}\Big/N\cos\frac{\Delta}{2} & N \text{ odd}\end{cases}$$

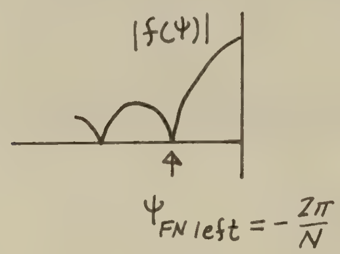
$\therefore\ |f(\pi-\Delta)|=|f(\pi+\Delta)|\Rightarrow|f(\psi)|$ symmetric about $\psi=\pi$.

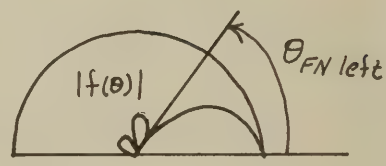
3.2-2

$$f(\psi) = \frac{\sin\psi}{2\sin\frac{\psi}{2}} \qquad (3\text{-}35)$$

$$\equiv \frac{2\sin(\frac{\psi}{2})\cos(\psi/2)}{2\sin(\psi/2)} = \underline{\underline{\cos\frac{\psi}{2}}} \quad \text{which is } (3\text{-}20)$$

3.2-3





At first null

$$\psi_{FN\,left} = \beta d\cos\theta_{FN\,left} + \alpha = \beta d(\cos\theta_{FN\,left} - 1) = -\frac{2\pi}{N}$$

So

$$1 - \cos\theta_{FN\,left} = \frac{2\pi}{N\beta d} = \frac{2\pi}{N\frac{2\pi}{\lambda}d} = \frac{\lambda}{Nd}$$

$$\approx 1 - \left(1 - \frac{\theta_{FN\,left}^2}{2}\right) = \frac{\theta_{FN\,left}^2}{2}$$

$$\therefore\ BWFN = 2\theta_{FN\,left} = \underline{\underline{2\sqrt{\frac{2\lambda}{Nd}}}} \quad \text{which is } (3\text{-}44).$$

3.2-4

At the half-power points

$$f(\psi_{HP}) = \frac{1}{\sqrt{2}} = \frac{\sin(\frac{N}{2}\psi_{HP})}{N\sin(\frac{1}{2}\psi_{HP})}$$

For $N=10$

$$\sin(5\psi_{HP}) = \frac{10}{\sqrt{2}}\sin\left(\frac{\psi_{HP}}{2}\right)$$

Solving by trial and error $5\psi_{HP} = \pm 1.39754$

$$\psi_{HP} = \pm 0.088975\pi = \pm 0.279523 = \beta d\cos\theta_{HP} \qquad (\alpha = 0)$$

$$\theta_{HP} = \cos^{-1}\left[\pm\frac{0.088975\pi}{2\pi d/\lambda}\right] = \cos^{-1}\left(\pm 0.0444875\frac{\lambda}{d}\right) \simeq 1 \mp 0.0444875\frac{\lambda}{d}$$

$$\theta_{HP\,right} = 1 - 0.0444875\frac{\lambda}{d}$$

$$\theta_{HP\,left} = 1 + 0.0444875\frac{\lambda}{d}$$

3.2-4 (con't)

$$HP = |\theta_{HP_{right}} - \theta_{HP_{left}}| = 2\left(0.0444875\frac{\lambda}{d}\right) = 0.088975\frac{\lambda}{d}$$

$$= \underbrace{0.88975}_{K}\frac{\lambda}{Nd} \qquad \text{for } N=10$$

For $N=20$

$$\psi_{HP} = 0.04434\pi = 0.1393 \qquad \text{by trial and error}$$

$$\theta_{HP} = \cos^{-1}\left[\pm\frac{0.04434\pi}{2\pi d/\lambda}\right] = \cos^{-1}\left[\pm\frac{0.04434}{2}\frac{\lambda}{d}\right]$$

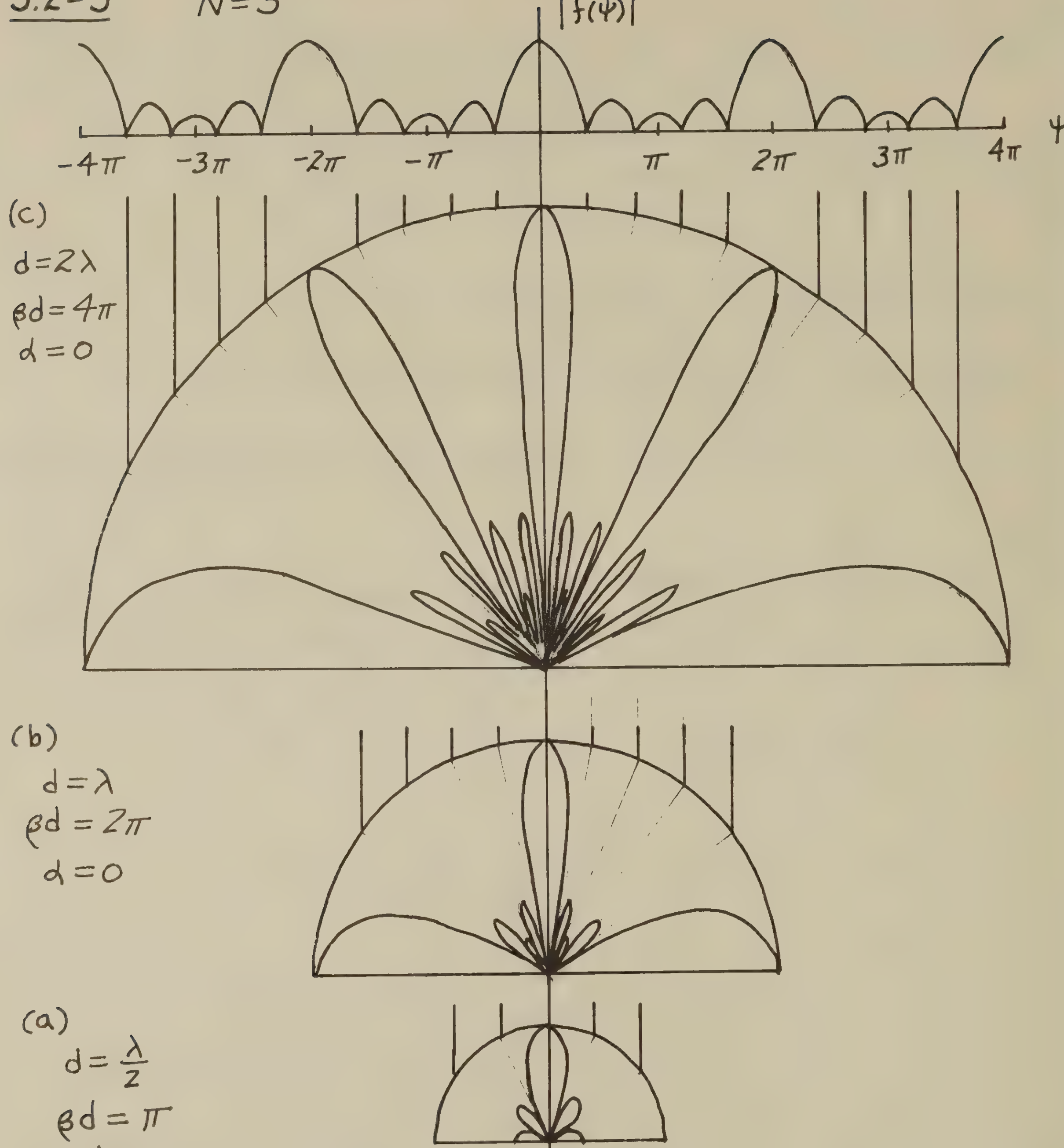
$$HP \approx 2\left(\frac{0.04434}{2}\frac{\lambda}{d}\right) = 0.04434\frac{\lambda}{d} = \underbrace{0.8868}_{K}\frac{\lambda}{Nd} \quad (N=20)$$

Thus

N	K
10	0.8898
20	0.8868

As N increases K converges toward 0.886 as in (3-45).

3.2-5 $N=5$



(d) $d=\frac{\lambda}{2}$

$\beta d=\pi$

$\theta_0=45^\circ$

$\alpha=-\beta d\cos 45^\circ$

$=-0.707\pi=-127.288^\circ$

(e) $d=\frac{\lambda}{2}$

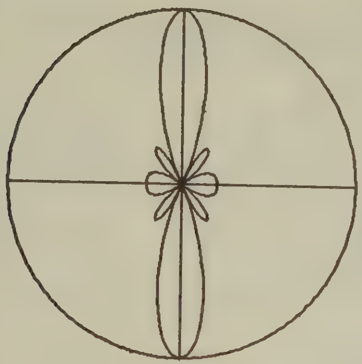
$\beta d=\pi$

$\theta_0=0^\circ$

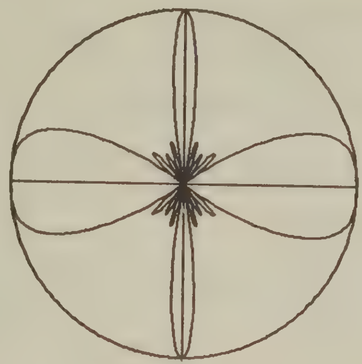
$\alpha=-\pi\cos 0^\circ$

$=-\pi=-180^\circ$

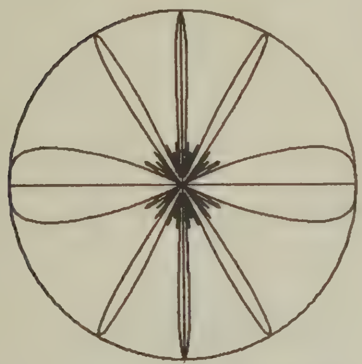
<u>3.2-6</u> ARRFAC and a pen and ink plotter version of PPLOT were used to produce these plots.



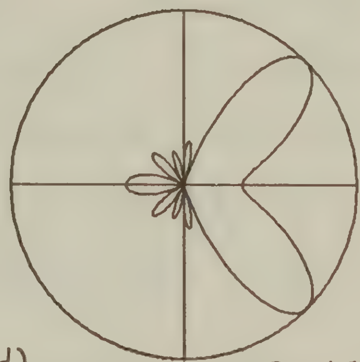
(a) $N=5$, $d=\lambda/2$, $\theta_0=90°$



(b) $N=5$, $d=\lambda$, $\theta_0=90°$



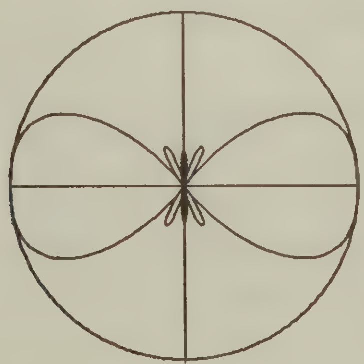
(c) $N=5$, $d=2\lambda$, $\theta_0=90°$



(d) $N=5$, $d=\lambda/2$, $\theta_0=45°$

(d) THE NUMBER OF ELEMENTS=N= 5
THE SPACINGS=D= 0.5000 WAVELENGTHS
THE DESIRED MAIN BEAM MAXIMUM DIRECTION=THETA0= 45.0000 DEGREES
THE REQUIRED PHASE SHIFT BETWEEN ELEMENTS=ALPHA= -127.2792 DEGREES

(e) $N=5$
$d=\lambda/2$
$\theta_0=0°$

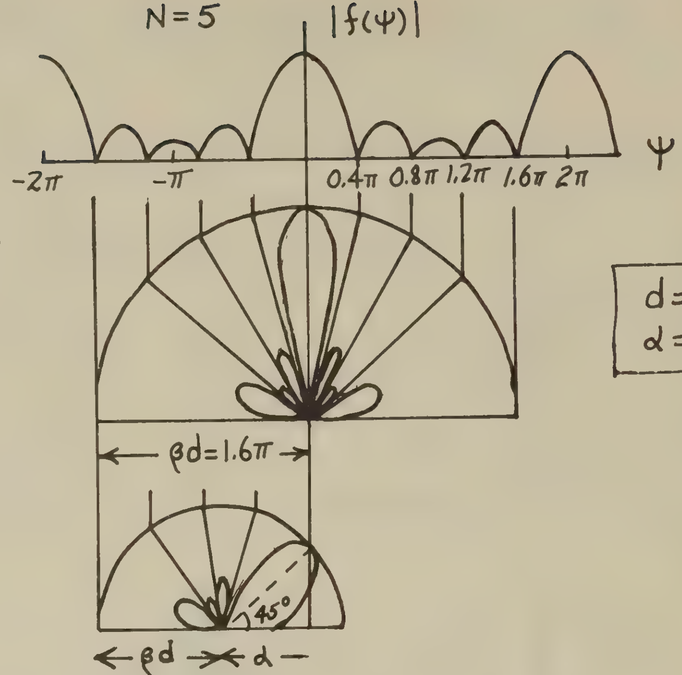


(e) THE NUMBER OF ELEMENTS=N= 5
THE SPACINGS=D= 0.5000 WAVELENGTHS
THE DESIRED MAIN BEAM MAXIMUM DIRECTION=THETA0= 0.0 DEGREES
THE REQUIRED PHASE SHIFT BETWEEN ELEMENTS=ALPHA= -180.0000 DEGREES

3.2-7

Beamwidth $\sim \frac{\lambda}{Nd} \sim \frac{\lambda}{N\beta d}$

So we make βd as large as possible for small beamwidth.



$$\boxed{\begin{array}{l} d = 0.8\lambda \\ \alpha = 0 \end{array}}$$

(a)

$\alpha = -\beta d \cos\theta_0 = 0$

$\beta d = 1.6\pi$

$d = \frac{1.6\pi}{2\pi/\lambda} = 0.8\lambda$

(b) Visible region

$\alpha - \beta d < \psi < \alpha + \beta d$

$-\beta d \cos\theta_0 - \beta d < \psi$

$< -\beta d \cos\theta_0 + \beta d$

$\alpha = -\beta d \cos\theta_0 = -\frac{\beta d}{\sqrt{2}}$ for $\theta_0 = 45°$

$\alpha - \beta d = -1.6\pi$ to come up to left grating lobe

Solving $\alpha = -\frac{\beta d}{\sqrt{2}}$ and $\alpha - \beta d = -1.6\pi$

$\beta d = 0.9373\pi \Rightarrow \boxed{d = 0.469\lambda}$ $\quad \alpha = \frac{-\beta d}{\sqrt{2}} = \boxed{-0.663\pi}$

3.2-8

First note that $d = 0.35\lambda$ satisfies (3-47)

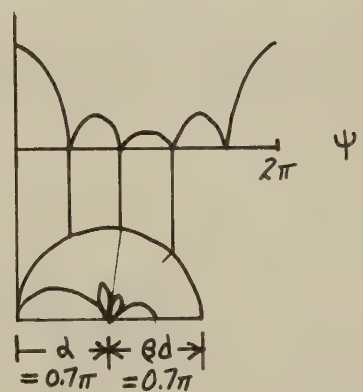
$2\beta d = 2\frac{2\pi}{\lambda}0.35\lambda = 1.4\pi$

$\leq 2\pi - \frac{\pi}{N} = 2\pi - \frac{\pi}{5} = 1.8\pi$

Now

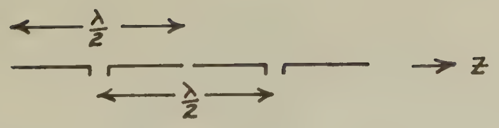
$\alpha = -\beta d \cos\theta_0 = -\frac{2\pi}{\lambda}0.35\lambda \cos 180°$

$\boxed{\alpha = 0.7\pi}$



Comparing to Fig. 3-14 we see, as claimed, the increased directivity pattern has a narrower main beam and higher side lobes.

3.3-1



From (2-9) $$g_a(\theta) = \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}$$

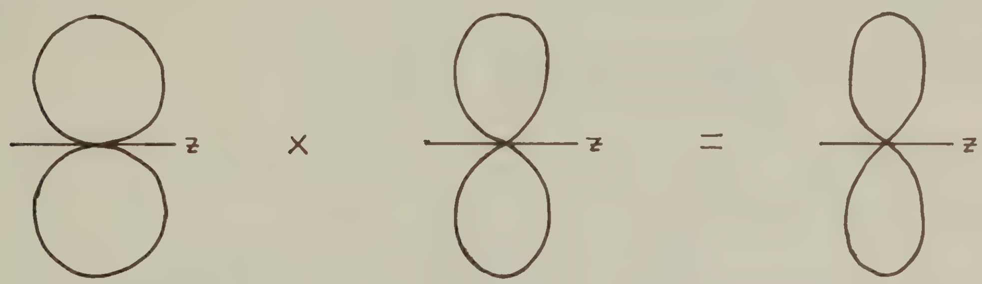
From (3-6) the array factor for a two element half-wavelength spaced, equally excited array

$$f(\theta) = \cos(\tfrac{\pi}{2}\cos\theta)$$

Thus

$$F(\theta) = g_a(\theta)\, f(\theta) = \boxed{\frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin\theta}}$$

$g_a(\theta) = \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}$ $\quad$ $f(\theta) = \cos(\frac{\pi}{2}\cos\theta)$ $\quad$ $F(\theta)$



(Fig. 2-5b) $\quad$ (Fig. 3-3b)

3.3-2

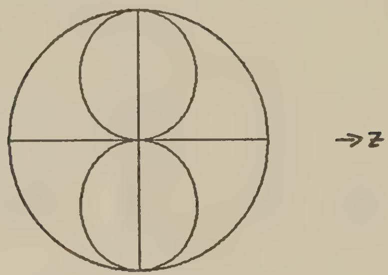
From (2-9),
$$g_a(\theta) = \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}$$

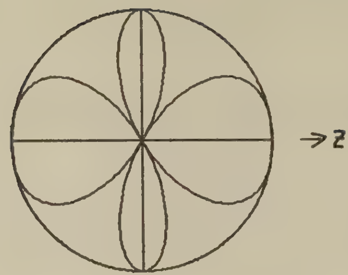
From (3-13) the array factor for a two element one-wavelength spaced, equally excited array

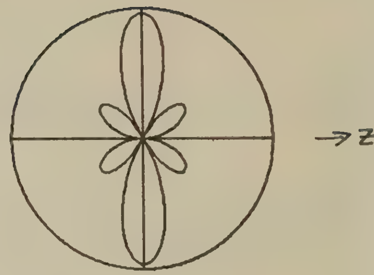
$$f(\theta) = \cos(\pi\cos\theta)$$

So
$$F(\theta) = g_a(\theta)\, f(\theta) = \boxed{\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\cos(\pi\cos\theta)}$$

3.3-2 (con't)



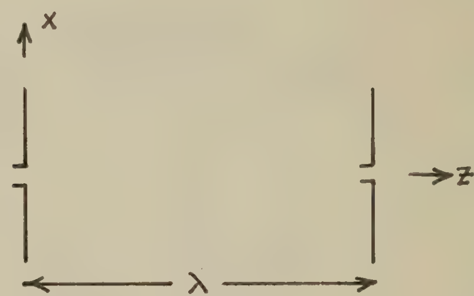




$g_a \quad \times \quad f \quad = \quad F$

(Fig. 2-5b) (Fig. 3-6c)

3.3-3



From (3-69) the $\lambda/2$-dipole pattern for this geometry is

$$g_a(\theta,\phi) = \frac{\cos\left(\frac{\pi}{2}\sin\theta\cos\phi\right)}{\sqrt{1-\sin^2\theta\cos^2\phi}}$$

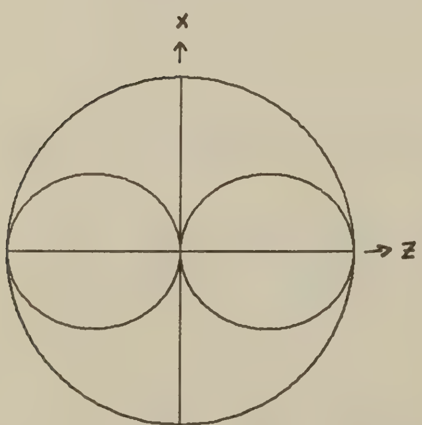
From (3-13) the array factor is

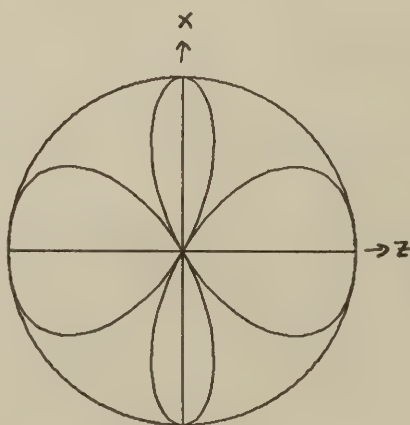
$$f(\theta) = \cos(\pi\cos\theta)$$

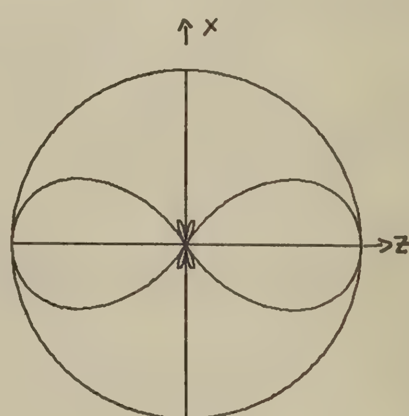
Thus

$$F(\theta,\phi) = g_a(\theta,\phi)\, f(\theta) = \frac{\cos\left(\frac{\pi}{2}\sin\theta\cos\phi\right)}{\sqrt{1-\sin^2\theta\cos^2\phi}}\cos(\pi\cos\theta)$$

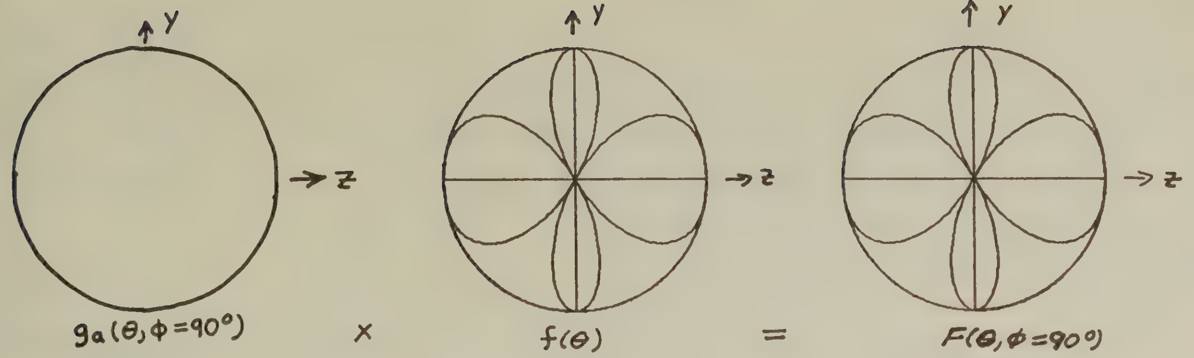
$g_a(\theta,\phi=0°) \quad \times \quad f(\theta) \quad = \quad F(\theta,\phi=0°)$



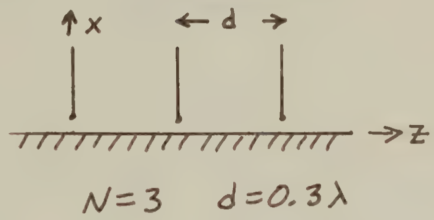




(Fig. 2-5b) (Fig. 3-6c)



3.3-4



(a) From (3-51) for a Hansen-Woodyard array

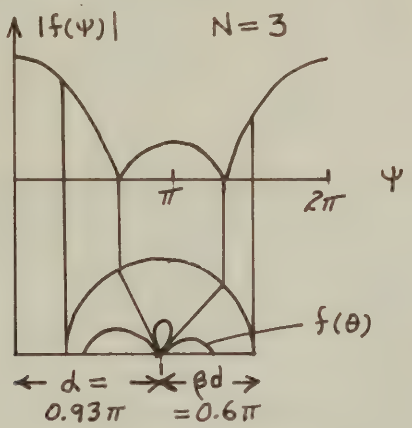
$$d < \frac{\lambda}{2}\left(1-\frac{1}{N}\right) = \frac{\lambda}{2}\left(1-\frac{1}{3}\right) = \frac{\lambda}{3}$$

The specification of $d = 0.3\lambda$ satisfies this.

Then from (3-49)

$$\alpha = \pm\left(\beta d + \frac{\pi}{N}\right) = \pm\left(\frac{2\pi}{\lambda}0.3\lambda + \frac{\pi}{3}\right) = \pm(0.6\pi + 0.33\pi) = \boxed{0.93\pi}$$ choosing + sign

(b)



(c)

From (3-69)

$$g_a(\theta,\phi) = \begin{cases} \dfrac{\cos\left(\frac{\pi}{2}\sin\theta\cos\phi\right)}{\sqrt{1-\sin^2\theta\cos^2\phi}} & \text{above ground plane} \\ 0 & \text{below ground plane} \end{cases}$$

From (3-17) and results of (a)

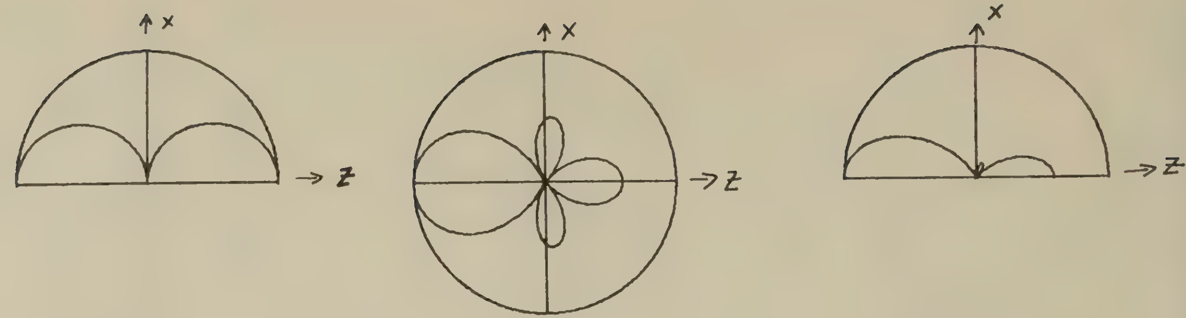
$$\psi = \beta d\cos\theta + \alpha = 0.6\pi\cos\theta + 0.93\pi$$

From (3-33) $\quad f(\psi) = \dfrac{\sin\left(\frac{N\psi}{2}\right)}{N\sin\psi/2}$

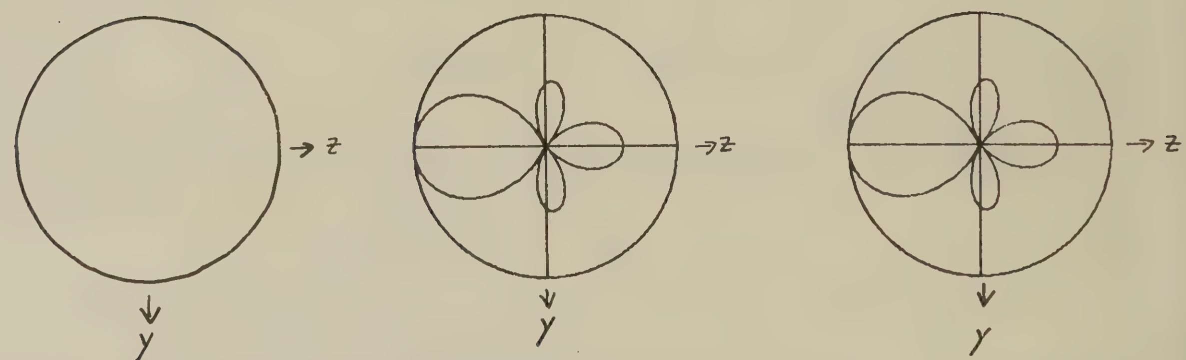
So

$$F(\theta,\phi) = g_a(\theta,\phi)\,f(\theta) = \frac{\cos\left(\frac{\pi}{2}\sin\theta\cos\phi\right)}{\sqrt{1-\sin^2\theta\cos^2\phi}}\;\frac{\sin[0.9\pi\cos\theta + 1.3995\pi]}{3\sin[0.3\pi\cos\theta + 0.4665\pi]}$$

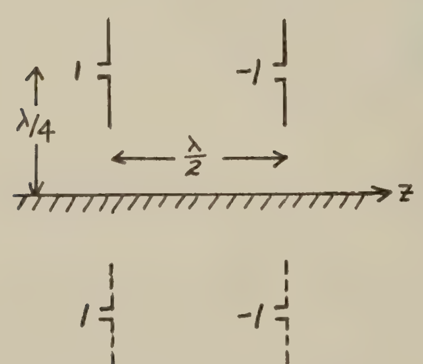
above the ground plane.



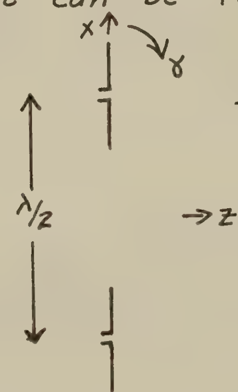
$g_a(\theta,\phi)$ X $f(\theta)$ = $F(\theta,\phi)$



<u>3.3-5</u> By method of images each element can be replaced by

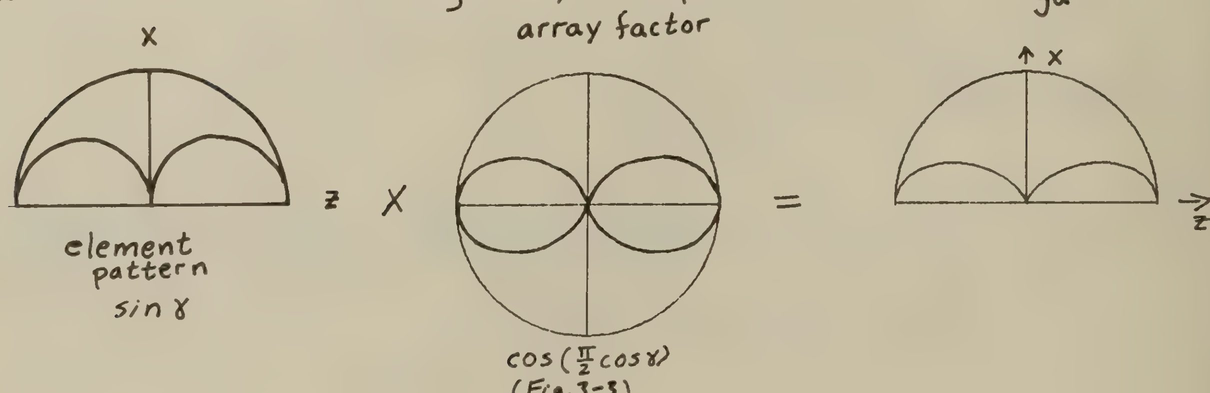


itself plus its image giving



$f(\gamma)=\cos(\frac{\pi}{2}\cos\gamma)$

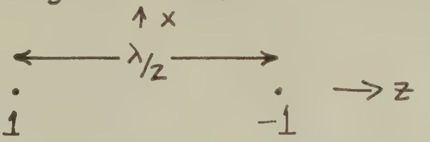
which has the following xz-plane pattern



3.3-5 (con't)

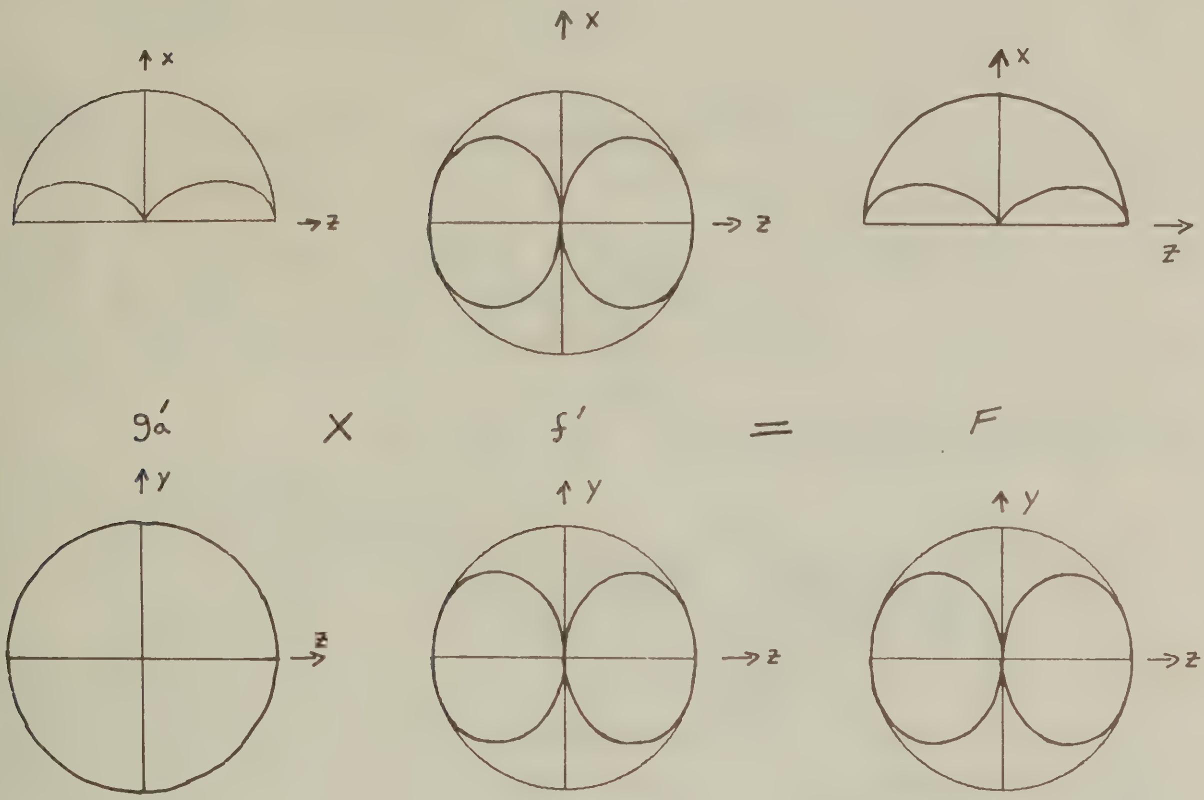
and the yz-plane pattern is isotropic.

Now arraying the right and left array halfs:



$$|f'| = |\sin(\tfrac{\pi}{2}\cos\theta)|$$

Thus the total pattern F is



3.3-6

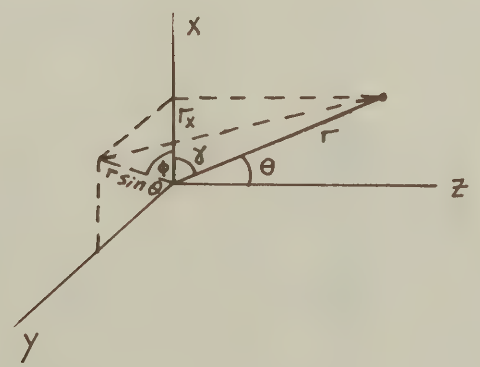
The projection of r along the x-axis can be found in two ways:

$r_x = r \cos\gamma$

$r_x = (r\sin\theta)\cos\phi$

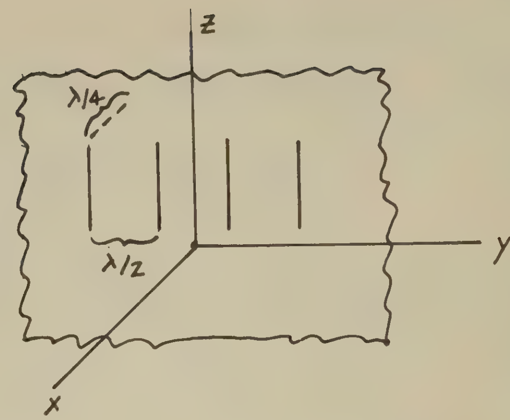
Equating gives

$$\boxed{\cos\gamma = \sin\theta\cos\phi}$$

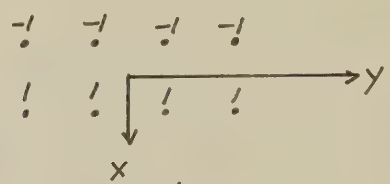


3.3-7

The image of this array is another array of four half-wave dipoles. Due to image theory the image dipoles are of opposite phase to the primary array.

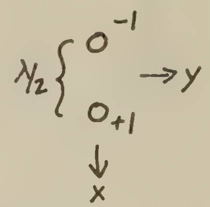


Top view



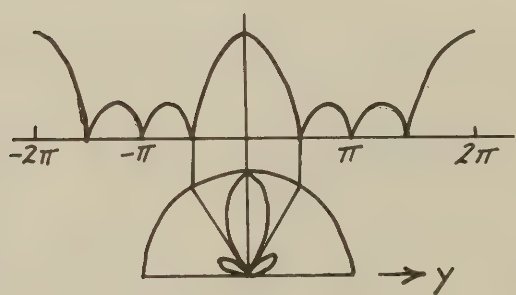
where • ≡ isotropic point source

This is equivalent to



where O ≡ array of 4 isotropic point sources.

For the 4-element array:

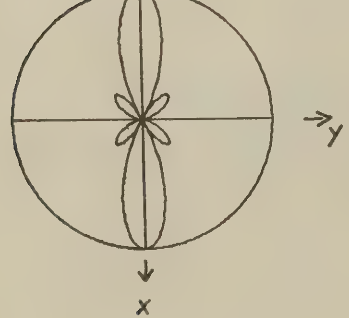


$$\beta d = \frac{2\pi}{\lambda}\frac{d}{2} = \pi$$

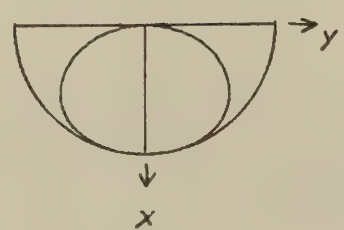
$$\alpha = 0$$

In the xy-plane

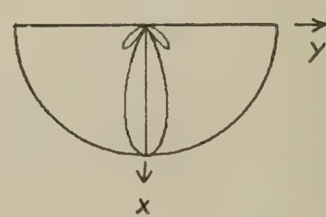
Element pattern (from above)	X	Array factor (see Fig. 3-4)	=	Total pattern F



4-element subarray

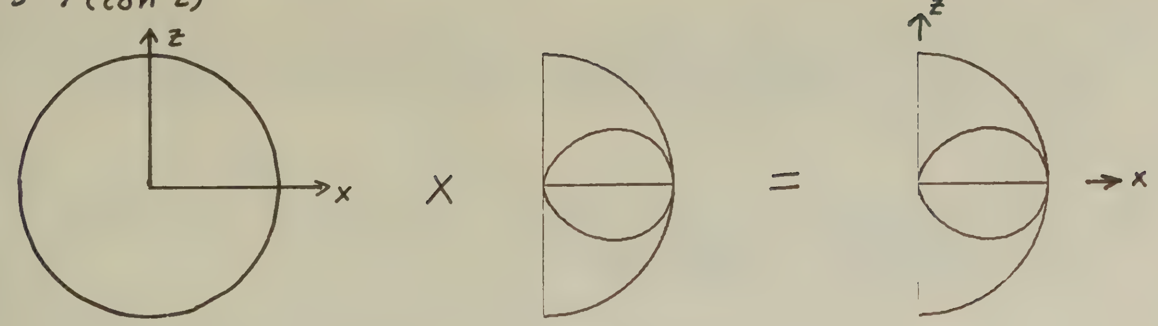


2-element subarray



8-element array

3.3-7 (con't)



4-element subarray | 2-element subarray | total pattern

<u>3.3-8</u>

The plots in Problem 3.3-1,3,4,5,7 solutions were obtained using the ARRPAT program and a pen & ink plotter version of PPLOT.

<u>3.3-9</u>

$$e^{j0} \quad e^{-j\frac{\pi}{2}} \quad e^{-j(\frac{\pi}{2}+\frac{2\pi}{5})} \quad e^{-j(\pi+\frac{2\pi}{5})}$$

$$\bullet \leftarrow \frac{\lambda}{4} \rightarrow \bullet\bullet \leftarrow \frac{\lambda}{5} \rightarrow \bullet\bullet \leftarrow \frac{\lambda}{4} \rightarrow \bullet$$

Consider this to be an array of two subarrays

$$e^{j0} \qquad e^{-j(\frac{\pi}{2})}$$

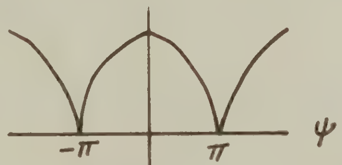
$$\bullet \leftarrow \frac{\lambda}{4} \rightarrow \bullet \qquad \text{(call this the "element")}$$

arrayed as

$$e^{-j\frac{\pi}{4}} \qquad e^{-j(\frac{\pi}{2}+\frac{2\pi}{5}+\frac{\pi}{4})} \qquad \text{or} \qquad e^{j0} \qquad e^{-j0.9\pi}$$

$$\text{o} \leftarrow 0.45\lambda \rightarrow \text{o} \qquad\qquad \text{o} \leftarrow 0.45\lambda \rightarrow \text{o}$$

The patterns of the subarrays are

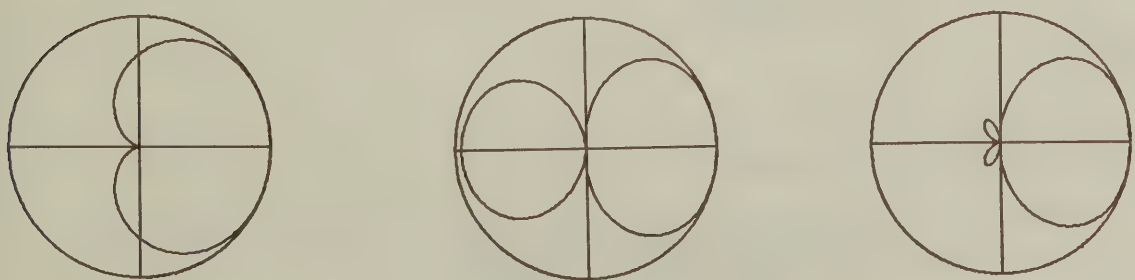


element pattern:  $\beta d = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$ $\quad \alpha = -\pi/2$

array factor:  $\beta d = \frac{2\pi}{\lambda} 0.45\lambda = 0.9\pi$ $\quad \alpha = -0.9\pi$

The total pattern:

element pattern $\quad$ X $\quad$ array factor $\quad$ = $\quad$ F

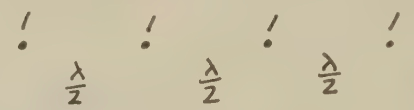


3.3-9 (con't)

Equation (3-33) and the graphical procedure cannot be used to verify the result since they rely on the elements being equally spaced.

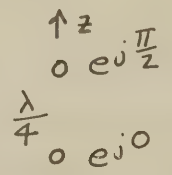
<u>3.3-10</u>

The eight element array can be decomposed into two subarrays : the "element"



with a pattern given in Prob. 3.3-7

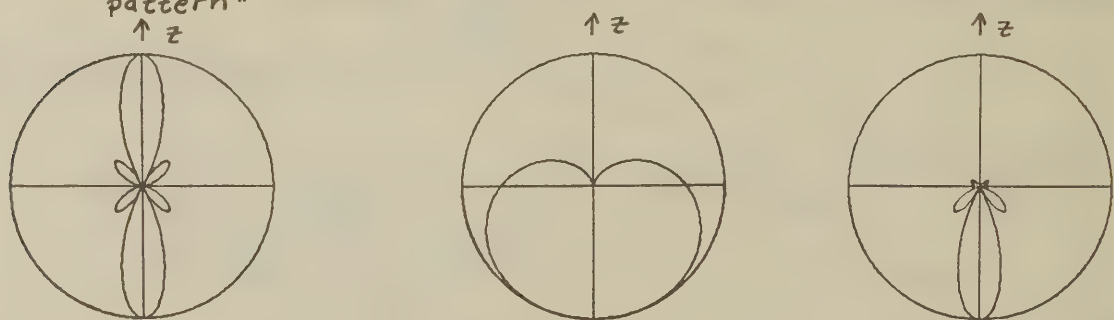
and an "array"



with a pattern given in Fig. 3-4

The total pattern the follows as

"element pattern" × "array factor" = Total pattern F



<u>3.4-1</u>

From (3-79) $D = N$ for $d = n\frac{\lambda}{2}$, $\alpha = 0$

(a) $N = 2$, $d = \lambda/2$

$D = N = 2$ $\quad D(\text{dB}) = 10 \log 2 = \boxed{3\text{ dB}}$

(b) $N = 10$, $d = \lambda/2$

$D = N = 10 = \boxed{10\text{ dB}}$

(c) $N = 15$, $d = \lambda$

$D = N = 15 = \boxed{11.8\text{ dB}}$

<u>3.4-2</u>

$d = 3\lambda/8 \quad N = 10$

(a) Broadside $\quad \alpha = 0 \quad \beta d = \frac{3}{4}\pi$

From (3-78)

$$D = \frac{N^2}{N + 2\sum_{m=1}^{N-1} \frac{N-m}{m\beta d}\sin(m\beta d)} \qquad N = 10, \ \beta d = \frac{3}{4}\pi$$

3.4-2 (con't)

The summation $= \sum_{m=1}^{9} \frac{10-m}{m\frac{3}{4}\pi} \sin m\frac{3}{4}\pi = \frac{4}{3\pi}[3.75] = 1.59$

$$D = \frac{10^2}{10+2(1.59)} = \boxed{7.59}$$

From (3-80) $D \approx 2\frac{L}{\lambda} = 2\frac{Nd}{\lambda} = 2\frac{10\frac{3}{8}\lambda}{\lambda} = \boxed{7.50}$

(b) Endfire $\alpha = \pm\beta d$

The sum in (3-78) is

$$\sum_{m=1}^{N-1} \frac{N-m}{m\beta d} \sin m\beta d \cos m\alpha = \frac{4}{3\pi} \sum_{m=1}^{9} \left(\frac{10}{m}-1\right) \underbrace{\sin m\tfrac{3}{4}\pi \cos m\tfrac{3}{4}\pi}_{\frac{1}{2}\sin m\frac{3}{2}\pi}$$

$$= \frac{4}{3\pi}\frac{1}{2}\left[9(-1)+0+\left(\tfrac{10}{3}-1\right)(1)+0+1(-1)+\left(\tfrac{10}{7}-1\right)(1)+0+\left(\tfrac{10}{9}-1\right)(-1)\right]$$

$$= \frac{2}{3\pi}[-7.3492] = -1.55955$$

So

$$D = \frac{10^2}{10+2(-1.55955)} = \boxed{14.53}$$

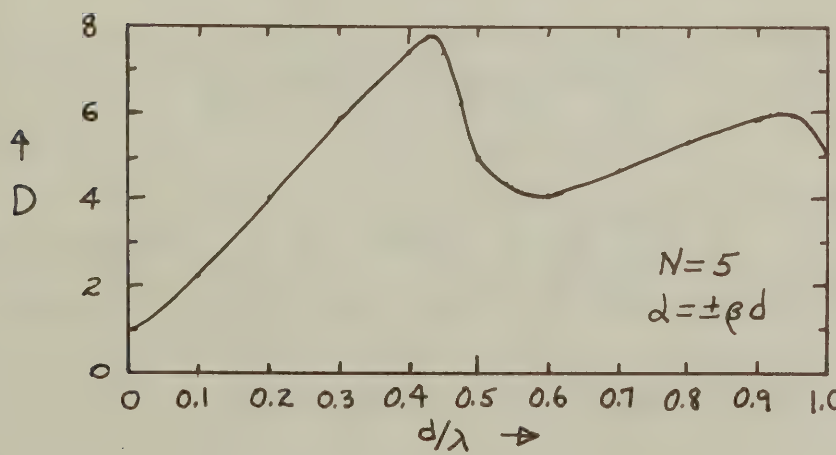
From (3-81) $D \approx 4\frac{L}{\lambda} = 4\frac{Nd}{\lambda} = 4\frac{10\frac{3}{8}\lambda}{\lambda} = \boxed{15.0}$

3.4-3

$N=5$ and endfire $(\alpha = \pm\beta d = \pm 2\pi\frac{d}{\lambda})$ in (3-78)

$$D = \left[\frac{1}{5} + \frac{2}{25}\sum_{m=1}^{4} \frac{5-m}{m2\pi\frac{d}{\lambda}} \sin(m2\pi\tfrac{d}{\lambda})\cos(m2\pi\tfrac{d}{\lambda})\right]^{-1}$$

$$= \left\{\frac{1}{5} + \frac{1}{50\pi}\frac{1}{d/\lambda}\left[4\sin(4\pi\tfrac{d}{\lambda}) + 1.5\sin(8\pi\tfrac{d}{\lambda}) + 0.667\sin(12\pi\tfrac{d}{\lambda}) + 0.25\sin(16\pi\tfrac{d}{\lambda})\right]\right\}^{-1}$$



3.4-4

(a)
$$D=\frac{4\pi}{\Omega_A}\approx\frac{4\pi}{2\pi HP}=\frac{2}{HP}\approx\frac{2}{0.886\frac{\lambda}{L}}=\boxed{2.26\frac{L}{\lambda}}$$

which compares to $D\approx 2\frac{L}{\lambda}$ of (3-80).

(b)
$$D=\frac{101}{HP_d-0.0027(HP_d)^2}=\frac{101}{0.886\frac{180}{\pi}\frac{\lambda}{L}-0.0027\left(0.886\frac{180}{\pi}\frac{\lambda}{L}\right)^2}$$
$$=\frac{101}{50.764\frac{\lambda}{L}-6.958\left(\frac{\lambda}{L}\right)^2}=\frac{101}{50.764-6.958\left(\frac{\lambda}{L}\right)}\frac{L}{\lambda}$$
$$\approx\frac{101}{50.764}\frac{L}{\lambda}\quad\text{for } L\gg\lambda$$

$\boxed{D=1.99\frac{L}{\lambda}}$ which compares very well to $D\approx 2\frac{L}{\lambda}$.

3.4-6

Ordinary endfire ; $d=\lambda/4$:

$\beta d=\frac{2\pi}{\lambda}\frac{\lambda}{4}=\frac{\pi}{2}$ and $\alpha=\pm\beta d=\pm\frac{\pi}{2}$

Then

$\sin m\beta d\cos m\alpha=\sin m\frac{\pi}{2}\cos m\frac{\pi}{2}=0$ for all m

Thus (3-78) reduces to

$$D=\frac{1}{\frac{1}{N}+0}=N\qquad\therefore\ \boxed{D=N}$$

3.4-5

$N=8$, $d=0.7\lambda$, broadside

(a) From Fig. 3-20 $D=11=\boxed{10.41\text{ dB}}$

(b) Using (3-80)
$$D\approx 2\frac{Nd}{\lambda}=2\frac{8(0.7\lambda)}{\lambda}=11.2=\boxed{10.49\text{ dB}}$$

3.5-2

(a) Eqn. (3-90) is
$$\Omega_A=\frac{2\pi}{\left(\sum_{k=0}^{N-1}A_k\right)^2}\sum_{m=0}^{N-1}\sum_{p=0}^{N-1}A_mA_p e^{j(\alpha_m-\alpha_p)}\underbrace{\int_0^{\pi}e^{j\beta(z_m-z_p)\cos\theta}\sin\theta\, d\theta}_{\mathcal{J}}$$

Let $v=\cos\theta$, then $dv=-\sin\theta\, d\theta$

3.5-2 (con't)

So $\mathcal{I} = \int_{1}^{-1} e^{j\beta(z_m - z_p)v}(-dv) = \int_{-1}^{1} e^{j\beta(z_m-z_p)v}\,dv$

$$= \frac{e^{j\beta(z_m-z_p)} - e^{-j\beta(z_m-z_p)}}{j\beta(z_m-z_p)} = 2\,\frac{\sin[\beta(z_m-z_p)]}{\beta(z_m-z_p)}$$

Using this in (3-90)

$$D = \frac{4\pi}{\Omega_A} = \frac{4\pi\left(\sum_{k=0}^{N-1} A_k\right)^2}{2\pi \sum_{m=0}^{N-1}\sum_{p=0}^{N-1} A_m A_p e^{j(\alpha_m-\alpha_p)}\, 2\,\frac{\sin[\beta(z_m-z_p)]}{\beta(z_m-z_p)}}$$

$$= \frac{\left(\sum_{k=0}^{N-1} A_k\right)^2}{\sum_{m=0}^{N-1}\sum_{p=0}^{N-1} A_m A_p e^{j(\alpha_m-\alpha_p)}\,\frac{\sin[\beta(z_m-z_p)]}{\beta(z_m-z_p)}}$$

which is (3-91).

(b) For $d = n\frac{\lambda}{2}$ $\qquad z_m - z_p = (m-p)\frac{\lambda}{2}$

$$\frac{\sin[\beta(z_m-z_p)]}{\beta(z_m-z_p)} = \frac{\sin\left[\frac{2\pi}{\lambda}(m-p)\frac{\lambda}{2}\right]}{\frac{2\pi}{\lambda}(m-p)\frac{\lambda}{2}} = \frac{\sin[(m-p)\pi]}{(m-p)\pi} = \begin{cases} 0 & m \neq p \\ 1 & m = p \end{cases}$$

So (3-91) becomes

$$D = \frac{\left(\sum_{k=0}^{N-1} A_k\right)^2}{\sum_{m=0}^{N-1} A_m^2} \qquad \text{which is (3-93).}$$

3.5-3

Since all arrays are of $d = \lambda/2$ and involve only isotropic elements, (3-93) applies

$$D = \frac{\left(\sum_{n=0}^{N-1} A_n\right)^2}{\sum_{n=0}^{N-1} A_n^2} = \frac{(2A_1 + 2A_2 + A_3)^2}{2A_1^2 + 2A_2^2 + A_3^2}$$

because the array excitations are symmetric:

$$\overset{\bullet}{A_1}\ \overset{\bullet}{A_2}\ \overset{\bullet}{A_3}\ \overset{\bullet}{A_4}\ \overset{\bullet}{A_5} \longrightarrow \overset{\bullet}{A_1}\ \overset{\bullet}{A_2}\ \overset{\bullet}{A_3}\ \overset{\bullet}{A_2}\ \overset{\bullet}{A_1}$$

3.5-3 (con't)

For Fig. 3-23c $A_1=1, A_2=4, A_3=6$

$$D=\frac{(2+8+6)^2}{2+32+36}=\boxed{3.657}$$

For Fig. 3-23d $A_1=1, A_2=1.61, A_3=1.94$

$$D=\frac{(2+3.22+1.94)^2}{2+2(1.61)^2+(1.94)^2}=\boxed{4.683}$$

For Fig. 3-23e $A_1=1, A_2=2.41, A_3=3.14$

$$D=\frac{(2+4.82+3.14)^2}{2+2(2.41)^2+(3.14)^2}=\boxed{4.2256}$$

For Fig. 3-25a $A_1=3, A_2=2, A_3=1$

$$D=\frac{(6+4+1)^2}{18+8+1}=\boxed{4.481}$$

3.6-1 $\quad Z_{11}=70\angle 0^\circ \qquad Z_{22}=100\angle 45^\circ \qquad Z_{12}=60\angle -10^\circ$

(a) Since antenna #2 is short circuited

$$Z_{1,in}=Z_{11}-\frac{(Z_{12})^2}{Z_{22}+Z_2}=Z_{11}-\frac{(Z_{12})^2}{Z_{22}} \quad \text{for } Z_2=0 \text{ in (3-102)}$$

$$=70\angle 0^\circ-\frac{(60\angle -10^\circ)^2}{100\angle 45^\circ}=54.8+j32.6=\boxed{63.8\angle 30.7^\circ}$$

(b) Since antenna #2 is open circuited $I_2=0$ and from (3-99)

$$V_2=Z_{21}I_1+Z_{22}I_2=Z_{21}I_1$$

And from (3-98)

$$V_1=Z_{11}I_1+Z_{12}I_2=Z_{11}I_1$$

So $\quad V_2/V_1\Big|_{I_2=0}=\dfrac{Z_{21}I_1}{Z_{11}I_1}=\dfrac{Z_{21}}{Z_{11}}$

$$V_2=\frac{Z_{21}}{Z_{11}}V_1=\frac{Z_{12}}{Z_{11}}V_1=\frac{60\angle -10^\circ}{70\angle 0^\circ}\,1\angle 0^\circ=\boxed{0.857\angle -10^\circ}$$

3.7-1

$d=0.4\lambda$ $\qquad -2d \quad -d \quad 0 \quad d \quad 2d \quad \rightarrow z$

$$\alpha_n=-\beta z_n\cos\theta_o=-\frac{2\pi}{\lambda}nd\cos\theta_o=-\frac{360^\circ}{\lambda}n\,0.4\lambda\cos\theta_o$$

$$=-n\,144^\circ\cos\theta_o$$

3.7-1 (con't)

For the center element at the origin

$\alpha_{-2} = 288° \cos\theta_0$, $\alpha_{-1} = 144° \cos\theta_0$, $\alpha_0 = 0$, $\alpha_1 = -144° \cos\theta_0$, $\alpha_2 = -288 \cos\theta_0$

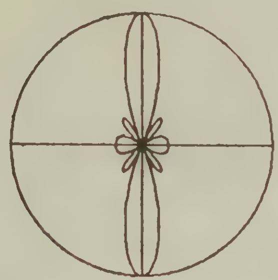
Case	θ_0	Fig. No.	α_{-2}	α_{-1}	α_0	α_1	α_2
Broadside	90°	3-30a	0	0	0	0	0
	75°	3-30b	74.5°	37.3°	0	−37.3°	−74.5°
	30°	3-30c	249.4°	124.7°	0	−124.7°	−249.4°
Endfire	0°	3-30d	288.0°	144.0°	0	−144.0°	−288.0°

3.7-2

(a) $d = 0.5\lambda$ $N = 5$

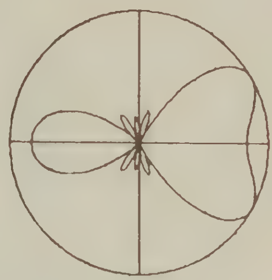
$\theta_0 = 90°$

$\alpha = 0°$



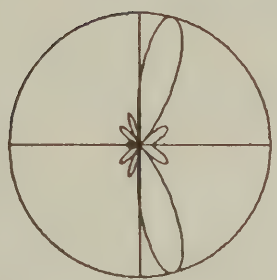
$\theta_0 = 30°$

$\alpha = -155.88°$



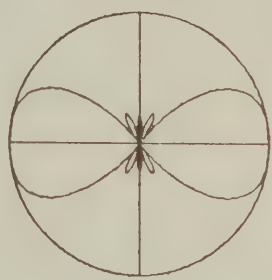
$\theta_0 = 75°$

$\alpha = -46.59°$



$\theta_0 = 0°$

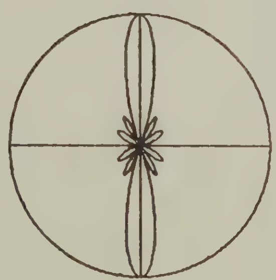
$\alpha = -180°$



(b) $d = 0.6\lambda$ $N = 5$

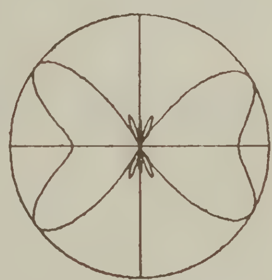
$\theta_0 = 90°$

$\alpha = 0°$



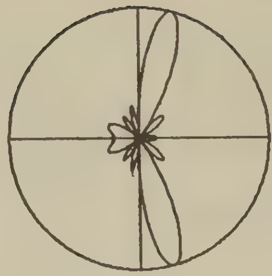
$\theta_0 = 30°$

$\alpha = -187.06°$



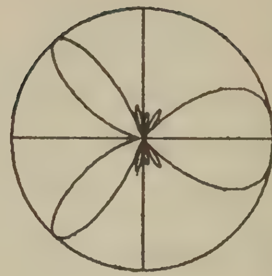
$\theta_0 = 75^\circ$

$\alpha = -55.9^\circ$



$\theta_0 = 0^\circ$

$\alpha = -216.0^\circ$



3.7-3

ARRPAT

THE ELEMENTS ARE COLLINEAR HALF-WAVE DIPOLES ON THE Z-AXIS

ELEMENT LOCATIONS, CURRENTS, AND PHASES

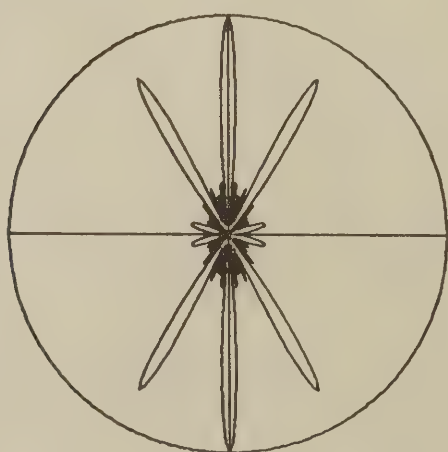
I	X(I)	Y(I)	Z(I)	A(I)	ALPHA(I)
1	0.0	0.0	0.0	1.0000	0.0
2	0.0	0.0	2.0000	1.0000	0.0
3	0.0	0.0	4.0000	1.0000	0.0
4	0.0	0.0	6.0000	1.0000	0.0
5	0.0	0.0	8.0000	1.0000	0.0

XZ-PLANE PATTERN PLOT

X-AXIS IS VERTICAL AND Z-AXIS IS HORIZONTAL

ANGLE IS THETA

THE PATTERN HAS BEEN DIVIDED BY EMAX= 5.000000



3.7–4

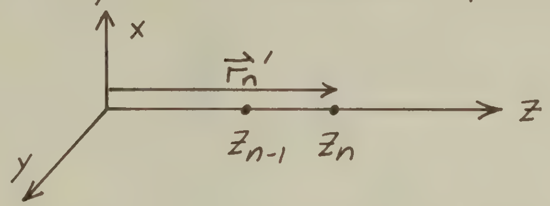
The phase term in (3–108) is $\beta(\hat{r}\cdot\vec{r}_n^{\,\prime})$

Using (A–4)

$$\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

For a linear array with element positions on the z-axis

$\vec{r}_n^{\,\prime} = z_n\hat{z}$



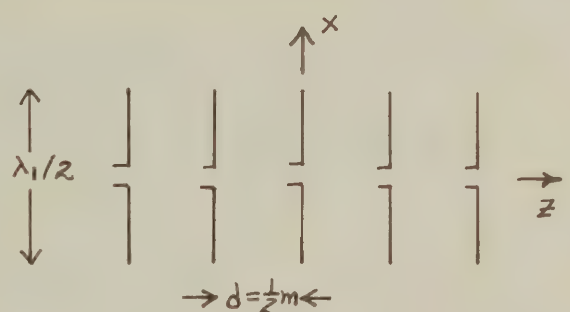
So

$$\beta\hat{r}\cdot\vec{r}_n^{\,\prime} = \beta(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta)\cdot z_n\hat{z}$$

$$= \boxed{\beta z_n\cos\theta}$$ which is the phase term in (3–65).

3.7–5

Since the array is to be parallel fed, the equal phase of excitation condition will be met at all frequencies due to equal line lengths.



At 300 MHz

$$\lambda_1 = \frac{c}{f_1} = \frac{3\times10^8}{3\times10^8} = 1m \text{, so } d_1 = \frac{\lambda_1}{2} = 0.5m \qquad \alpha_1 = 0$$

$\therefore d = 0.5m$

At 360 MHz

$$\lambda_2 = \frac{3\times10^8}{3.6\times10^8} = 0.833m\text{, so } d_2 = 0.5m = \frac{0.5}{0.833}(0.833m) = 0.6\lambda_2$$

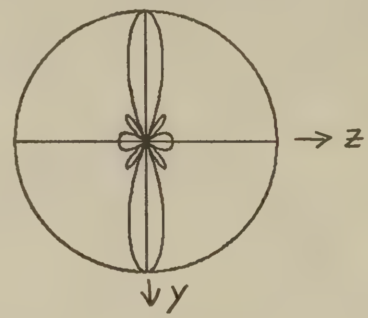
$$\alpha_2 = 0$$

In the yz-plane the element pattern is isotropic so the complete pattern in that plane is simply the array factor. Below are plotted the two patterns for each frequency. Note that for this 20% bandwidth the pattern is very stable.

3.7-5 (con't)

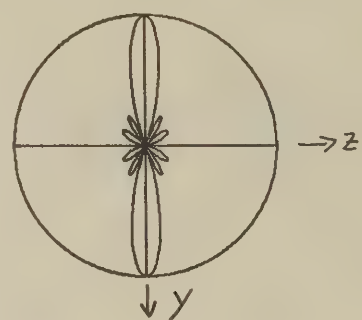
```
THE NUMBER OF ELEMENTS=N=     5
THE SPACINGS=D=     0.5000 WAVELENGTHS
THE DESIRED MAIN BEAM MAXIMUM DIRECTION=THETAO=    90.0000 DEGREES
THE REQUIRED PHASE SHIFT BETWEEN ELEMENTS=ALPHA =     0.0     DEGREES
```

$f = 300$ MHz

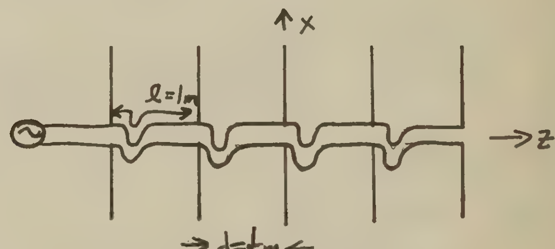



```
THE NUMBER OF ELEMENTS=N=     5
THE SPACINGS=D=     0.6000 WAVELENGTHS
THE DESIRED MAIN BEAM MAXIMUM DIRECTION=THETAO=    90.0000 DEGREES
THE REQUIRED PHASE SHIFT BETWEEN ELEMENTS=ALPHA =     0.0     DEGREES
```

$f = 360$ MHz



3.7-6



$N=5 \quad \{A_i\}=1$

At 300 MHz $\lambda_1 = 1m$

$d_1 = \frac{1}{2}m = \frac{\lambda_1}{2}$

$\alpha = -\beta_1 l$ where l = length of transmission line between each element. Assume phase constant is that of free space.

$= -\frac{2\pi}{\lambda_1} 1m = -2\pi \Rightarrow$ elements in phase

Hence pattern is same as that shown in Prob 3.7-5 for 300 MHz.

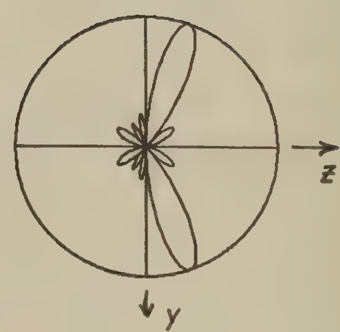
At 360 MHz

$d_2 = 0.6\lambda_2$

$\alpha_2 = -\beta_2 l = -\frac{2\pi}{\lambda_2}(1m) = -\frac{2\pi}{0.833m}(1m)$

$= -2.4\pi \rightarrow -0.4\pi = -72°$

The pattern is shown at the right.



CHAPTER 4

4.1-1

First $\cos(\alpha + \frac{\pi}{2}) = -\sin\alpha$ $\qquad$ $\cos\left[\pm(\alpha - \frac{\pi}{2})\right] = \sin\alpha$

So $\alpha + \frac{\pi}{2} = \cos^{-1}(-\sin\alpha)$ $\qquad$ $-(\alpha - \frac{\pi}{2}) = \cos^{-1}(\sin\alpha)$

Let $x = \sin\alpha$ or $\alpha = \sin^{-1}x$

Then

$$\cos^{-1}(-x) = \sin^{-1}x + \frac{\pi}{2}$$

$$\cos^{-1}(x) = -(\sin^{-1}x - \frac{\pi}{2})$$

Subtracting

$$\cos^{-1}(-x) - \cos^{-1}(x) = 2\sin^{-1}x \qquad \text{QED}$$

4.1-2

From (4-15) $\quad HP = 2\cos^{-1}(1 - 0.443\frac{\lambda}{L})$

let $\alpha = \frac{HP}{2} = \cos^{-1}(1-y)$ where $y = 0.443\frac{\lambda}{L}$

or $\cos\alpha = 1 - y$

And

$$\cos^2\alpha = 1 - 2y + y^2 \simeq 1 - 2y \qquad \text{since } y \ll 1$$

$$= \cos^2\alpha + \sin^2\alpha - 2y$$

So $\sin^2\alpha = 2y$ or $\sin\alpha = \sqrt{2y}$

But $\sin\alpha \simeq \alpha$ for $\alpha \ll 1$

Thus $\alpha \approx \sqrt{2y}$ or $\frac{HP}{2} = \sqrt{2(0.443\frac{\lambda}{L})}$

$\therefore$ $\boxed{HP \simeq 2\sqrt{0.886\frac{\lambda}{L}}}$ which is (4-16).

4.1-3

From (4-8) for a uniform line source

$$E_\theta = \frac{j\omega\mu e^{-j\beta r}}{4\pi r} I_o L \sin\theta \frac{\sin u}{u} \quad \text{where } u = \frac{\beta L}{2}(\cos\theta - \cos\theta_o) \quad (4\text{-}17)$$

For $\theta_o = 90°$ $\quad u = \frac{\beta L}{2}\cos\theta$

when $L \ll \lambda$ $\quad u = \frac{2\pi}{\lambda}\frac{L}{2}\cos\theta < \pi\frac{L}{\lambda} \ll \pi$. Then $\frac{\sin u}{u} \simeq 1$

And $\boxed{E_\theta \simeq \frac{j\omega\mu e^{-j\beta r}}{4\pi r} I_o L \sin\theta}$ which is the same as the ideal dipole expression (1-71).

4.1-4

(a) 8λ, uniform line source

From (4-14) $HP = 0.886 \frac{\lambda}{L} = 0.886 \frac{\lambda}{8\lambda} = 0.11075 = \boxed{6.34^\circ}$

From (4-21) $D_u = 2\frac{L}{\lambda} = 2\frac{8\lambda}{\lambda} = 16 = \boxed{12.04\text{ dB}}$

(b) 8λ, endfire uniform line source

From (4-16) $HP = 2\sqrt{0.886\frac{\lambda}{L}} = 2\sqrt{0.11075} = 0.6656 = \boxed{38.14^\circ}$

From (4-22) $D_u = 4\frac{L}{\lambda} = 4\frac{8\lambda}{\lambda} = 32 = \boxed{15.05\text{ dB}}$

(c) 16λ, broadside uniform line source

$HP = 0.886\frac{\lambda}{L} = 0.886\frac{\lambda}{16\lambda} = 0.055375\text{ r} = \boxed{3.17^\circ}$

$D_u = 2\frac{L}{\lambda} = 2\frac{16\lambda}{\lambda} = 32 = \boxed{15.05\text{ dB}}$

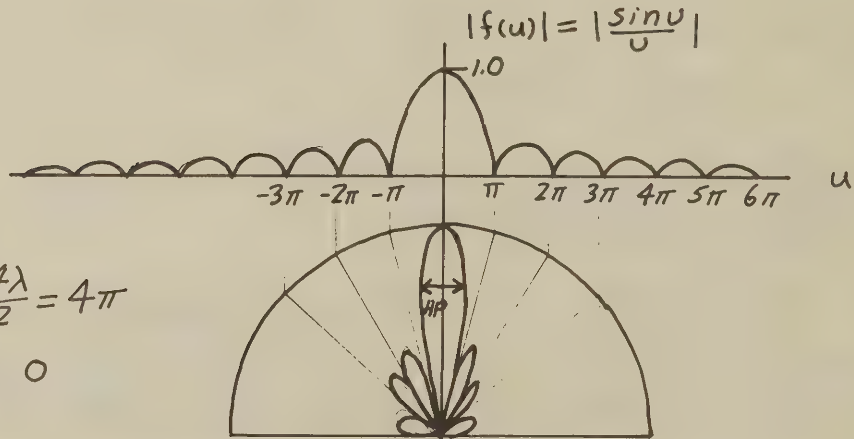
(d) 16λ, endfire

$HP = 2\sqrt{0.886\frac{L}{\lambda}} = 2\sqrt{0.886\frac{\lambda}{16\lambda}} = 2\sqrt{0.055735} = 0.04706 = \boxed{26.97^\circ}$

$D_u = 4\frac{L}{\lambda} = 4\frac{16\lambda}{\lambda} = 64 = \boxed{18.06\text{ dB}}$

4.1-5

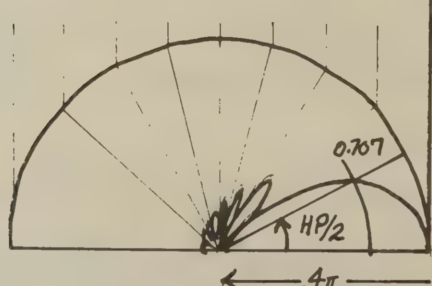
(a)



Broadside:

$\frac{\beta L}{2} = \frac{2\pi}{\lambda}\frac{4\lambda}{2} = 4\pi$

$-\frac{\beta L}{2}\cos\theta_o = 0$



Endfire:

$\frac{\beta L}{2} = 4\pi$

$-\frac{\beta L}{2}\cos\theta_o = -4\pi\cos 0^\circ$

$= -4\pi$

4.1-5 (con't)

(b) Measuring the HP's

$HP(\text{broadside}) \approx \underline{16°}$ $\qquad HP(\text{end fire}) \approx \underline{60°}$

This apprach is very approximate.

(c)

$$HP(\text{broadside}) = 0.886\frac{\lambda}{L} = 0.886\frac{\lambda}{4\lambda} = 0.2215\ r = \underline{12.7°}$$

$$HP(\text{endfire}) = 2\sqrt{0.886\frac{\lambda}{L}} = 2\sqrt{0.2215} = 0.9413\ r = \underline{53.9°}$$

4.1-6

For broadside uniform line sources $u = \frac{\beta L}{2}\cos\theta$.

(a) 2λ

i) $\theta_{HP} = (90° - \frac{HP}{2}) = (90 - \frac{24.766}{2}) = 77.617$

$u_{HP} = \frac{2\pi}{\lambda}\frac{2\lambda}{2}\cos(77.617°) = 0.4289\pi$

$\sin\theta_{HP}\frac{\sin u_{HP}}{u_{HP}} = 0.7069$ ✓

ii) $\theta_{HP} = (90° - \frac{25.591}{2}) = 77.2045°$ $\qquad u_{HP} = 2\pi\cos(77.2045°) = 1.39155$

$\frac{\sin u_{HP}}{u_{HP}} = 0.7071$ ✓

iii) $HP = 0.886\frac{\lambda}{L} = 0.886\frac{\lambda}{2\lambda} = 0.443\ r = 25.382°$ ✓

(b) 5λ

i) $\theta_{HP} = (90° - \frac{HP}{2}) = (90 - \frac{10.112}{2}) = 84.944°$

$u_{HP} = \frac{2\pi}{\lambda}\frac{5\lambda}{2}\cos(84.944°) = 1.3843$; $\sin\theta_{HP}\frac{\sin u_{HP}}{u_{HP}} = 0.70709$ ✓

ii) $\theta_{HP} = (90° - \frac{10.166}{2}) = 84.917$ $\qquad u_{HP} = 5\pi\cos(84.917) = 1.3917$

$\frac{\sin u_{HP}}{u_{HP}} = 0.70705$ ✓

iii) $HP = 0.886\frac{\lambda}{L} = 0.886\frac{\lambda}{5\lambda} = 0.1772\ r = 10.153°$ ✓

(c) 10λ

i) $\theta_{HP} = (90° - \frac{5.071°}{2}) = 87.4645°$ $\qquad u_{HP} = \frac{2\pi}{\lambda}\frac{10\lambda}{2}\cos(87.4645°) = 1.38979$

$\sin\theta_{HP}\frac{\sin u_{HP}}{u_{HP}} = 0.70708$ ✓

ii) $\theta_{HP} = (90° - \frac{5.080°}{2}) = 87.46°$ $\qquad u_{HP} = 10\pi\cos(87.46°) = 1.39225$

$\frac{\sin u_{HP}}{u_{HP}} = 0.7068$ ✓

iii) $HP = 0.886\frac{\lambda}{10\lambda} = 0.0886\ r = 5.076°$ ✓

4.1-7

(a) From (4-20)

$$\Omega_A = 2\frac{\lambda}{L}\,\mathcal{J} \quad \text{where } \mathcal{J} = \int_{-a/2}^{b/2} \frac{\sin^2 u}{u^2}\,du \qquad a = (\beta-\beta_0)L \quad b = (\beta+\beta_0)L$$

And $D = 4\pi/\Omega_A$, so $\Omega_A = 4\pi/D$. Thus

$$\frac{4\pi}{D_u} = 2\frac{\lambda}{L}\mathcal{J} \quad \text{or} \quad \mathcal{J} = \frac{4\pi/D_u}{2\lambda/L} = \frac{\frac{2\pi}{\lambda}L}{D_u} = \frac{\beta L}{D_u} \quad \text{or} \quad \mathcal{J} = \frac{\beta L}{D_u}$$

Evaluating $\mathcal{J}$:

$$\mathcal{J} = \int_{-a/2}^{b/2} \frac{\sin^2 u}{u^2}du = \frac{1}{2}\int_{-a/2}^{b/2}\frac{1-\cos 2u}{u^2}du = \frac{1}{2}\left[\int_{-a/2}^{b/2}\frac{du}{u^2} - \int_{-a/2}^{b/2}\frac{\cos 2u}{u^2}du\right]$$

$$= \frac{1}{2}\left[-\frac{1}{u}\Big|_{-a/2}^{b/2} - \int_{-a}^{b}\frac{\cos v}{\frac{1}{4}v^2}\frac{dv}{2}\right] \quad \text{where } v = 2u,\ du = 2\,dv$$

$$= \frac{1}{2}\left[-\frac{2}{b} - \frac{2}{a}\right] - \int_{-a}^{b}\frac{\cos v}{v^2}dv = -\frac{1}{a} - \frac{1}{b} - \mathcal{J}'$$

$$\mathcal{J}' = \int_{-a}^{b}\frac{\cos v}{v^2}dv = -\frac{\cos v}{v}\Big|_{-a}^{b} - \int_{-a}^{b}\frac{\sin v}{v}dv$$

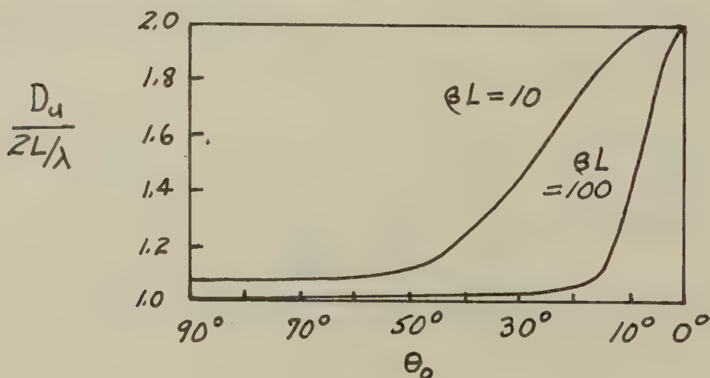
integrating by parts
$\int x\,dy = xy - \int y\,dx$
$x = \cos v \Rightarrow dx = -\sin v\,dv$
$dy = \frac{dv}{v^2} \Rightarrow y = -\frac{1}{v}$

$$= -\frac{\cos b}{b} + \frac{\cos a}{-a} - \underbrace{\int_0^b \frac{\sin v}{v}dv}_{Si(b)} + \underbrace{\int_0^{-a}\frac{\sin v}{v}dv}_{Si(-a)}$$

Thus
$$\mathcal{J} = -\frac{1}{a} + \frac{\cos a}{a} + \frac{\cos b}{b} - \frac{1}{b} - Si(a) - Si(b)$$

$$\therefore \frac{\beta L}{D_u} = \frac{\cos a - 1}{a} + \frac{\cos b - 1}{b} + Si(a) + Si(b) \qquad \text{Q.E.D.}$$

(b)
$$\frac{D_u}{2L/\lambda} = \pi\frac{D_u}{\beta L} = \frac{\pi}{\beta L/D_u} = \frac{\pi}{\frac{\cos a - 1}{a} + \frac{\cos b - 1}{b} + Si(a) + Si(b)}$$



(Hansen vol. I)

4.1-7 (con't)

(c) At broadside $\theta_0 = 90°$, $\cos\theta_0 = 0$ and $a = b = \beta L$

From (a) $$\frac{\beta L}{D_u} = 2\,\frac{\cos\beta L - 1}{\beta L} + 2\,Si(\beta L)$$

(d) As L becomes much larger than a wavelength (c) becomes

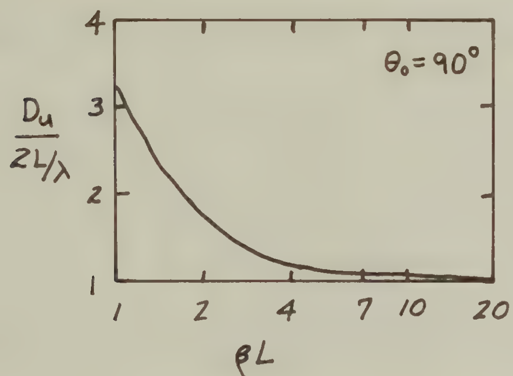
$$\frac{\cos\beta L - 1}{\beta L} \xrightarrow[\beta L\to\infty]{} 0 \qquad Si(\beta L) \xrightarrow[\beta L\to\infty]{} \int_0^\infty \frac{\sin u}{u}\,du = \frac{\pi}{2}$$

so $$\frac{\beta L}{D_u} \xrightarrow[\beta L\to\infty]{} 2(0) + 2\left(\frac{\pi}{2}\right) = \pi \Rightarrow D_u = \frac{2L}{\lambda} \text{ for } L \gg \lambda$$

which is (4-21).

(e) In the broadside case

$$\frac{D_u}{2L/\lambda} = \pi\,\frac{D_u}{\beta L} = \frac{\pi}{\beta L/D_u} = \frac{\pi/2}{\dfrac{\cos\beta L - 1}{\beta L} + Si(\beta L)}$$



(Hansen, vol. I)

4.2-1

Cosine-tapered broadside line source:

(a) Proceeding as in Section 4.1

$HP = 2\sin^{-1}\left(\frac{1.19}{2}\frac{\lambda}{L}\right)$, so $\frac{2}{\beta L}u_{HP} = \frac{1.19}{2}\frac{\lambda}{L}$, or $u_{HP} = \frac{2\pi}{\lambda}\frac{L}{2}\frac{1.19}{2}\frac{\lambda}{L} = 1.869$

Inserting this in (4-27)

$$f(u_{HP}) = \frac{\cos(1.869)}{1-\left(\frac{2}{\pi}1.869\right)^2} = 0.7067 \simeq \frac{1}{\sqrt{2}} \checkmark$$

(b) $$0 = \left.\frac{d f(u)}{du}\right|_{u=u_{SL}} = \frac{d}{du}\left(\frac{\cos u}{1-\left(\frac{2}{\pi}u\right)^2}\right)_{u=u_{SL}} = \left[\frac{-\sin u}{1-\left(\frac{2}{\pi}u\right)^2} + \frac{\cos u(-1)\left(-2\left(\frac{2}{\pi}\right)^2 u\right)}{\left(1-\left(\frac{2}{\pi}u\right)^2\right)^2}\right]_{u=u_{SL}}$$

or $0 = \left[1-\left(\frac{2}{\pi}u_{SL}\right)^2\right]\tan u_{SL} - \frac{8}{\pi^2}u_{SL}$; solving by trial and error $u_{SL} = 1.8895\pi$

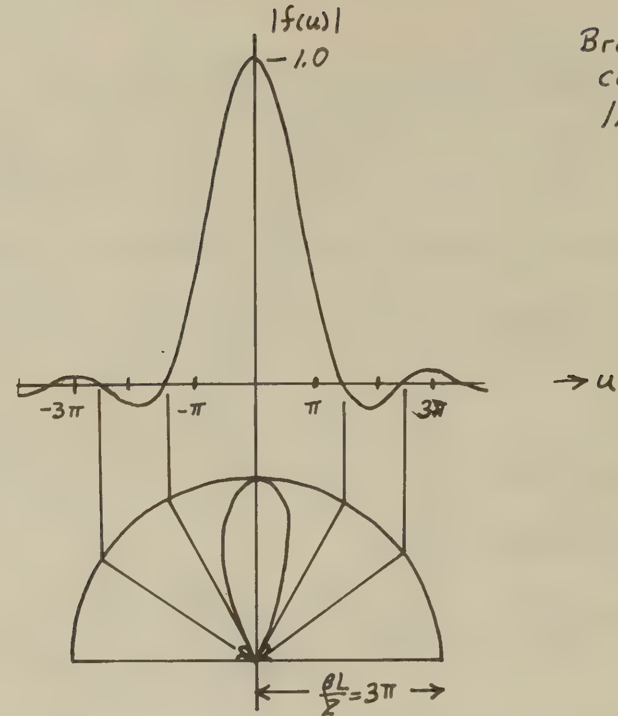
Then $f(1.8895\pi) = \frac{\cos(1.8895\pi)}{1-(2\cdot 1.8895)^2} = -0.070805 \longrightarrow -22.999\,\text{dB} \checkmark$

4.2-2 $L = 3\lambda$

$\frac{\beta L}{2} = \frac{2\pi}{\lambda}\frac{3\lambda}{2}$

$= 3\pi$

Broadside cosine-tapered line source



4.2-3

$L = 3$ m $\lambda = 0.3$ m @ 1GHz   cosine-squared tapered, broadside

(a) From Table 4-2

$$HP = 1.44\frac{\lambda}{L} = 1.44\frac{0.3}{3} = 0.144 \text{ rad} = \boxed{8.25°}$$

(b) From Table 4-2

$$D = 0.667 D_u = 0.667 \cdot 2\frac{L}{\lambda} = 0.667 \cdot 2\frac{3\lambda}{\lambda} = 13.34 = \boxed{11.25 \text{ dB}}$$

4.2-4

10λ line sources with following current distributions:

(a) Uniform

From (4-14) $HP \simeq 0.886\frac{\lambda}{L} = 0.886\frac{\lambda}{10\lambda} = 0.0886 \text{ rad} = \boxed{5.08°}$

From (4-21) $D_u = 2\frac{L}{\lambda} = 2\frac{10\lambda}{2} = 20 = \boxed{13 \text{ dB}}$

(b) Triangular (remaining relationships from Table 4-2)

$HP = 1.28\frac{\lambda}{L} = 0.128 \text{ rad} = \boxed{7.33°}$

$D = \frac{D}{D_u}D_u = 0.75(20) = 15 = \boxed{11.76 \text{ dB}}$

(c) Cosine

$HP = 1.19\frac{\lambda}{L} = 0.119 \text{ rad} = \boxed{6.82°}$

$D = 0.810 D_u = 0.810(20) = 16.2 = \boxed{12.1 \text{ dB}}$

(d) Cosine squared

$HP = 1.44\frac{\lambda}{L} = 0.144 \text{ rad} = \boxed{8.25°}$

$D = 0.667 D_u = 0.667(20) = 13.3 = \boxed{11.25 \text{ dB}}$

4.2-4 (con't)

(e) Cosine on a −10 dB pedestal

$$HP = 1.03\frac{\lambda}{L} = 0.103 = \boxed{5.90^\circ}$$

$$D = 0.92\,D_u = 0.92\,(20) = 18.4 = \boxed{12.65\,\text{dB}}$$

<u>4.2-5</u> Triangular current-tapered line source

(a) From Table 4-2a $\quad I(z) = 1 - \frac{2}{L}|z| \qquad |z| \le \frac{L}{2}$

$$f_{un}(u) = \int_{-\frac{L}{2}}^{\frac{L}{2}} I(z')\,e^{jcz'}\,dz' \qquad c = \beta\cos\theta + \beta_0$$

$$= \int_{-\frac{L}{2}}^{0} \left(1 + \frac{2}{L}z'\right)e^{jcz'}dz' + \int_{0}^{\frac{L}{2}}\left(1 - \frac{2}{L}z'\right)e^{jcz'}dz'$$

$$= \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{jcz'}dz' + \frac{2}{L}\left\{\int_{-\frac{L}{2}}^{0} z'e^{jcz'}dz' - \int_{0}^{\frac{L}{2}} z'e^{jcz'}dz'\right\}$$

$$= \frac{e^{jc\frac{L}{2}} - e^{-jc\frac{L}{2}}}{jc} - \frac{2}{L}\int_0^{\frac{L}{2}} z'\left(e^{-jcz'} + e^{jcz'}\right)dz'$$

$$= 2\,\frac{\sin\frac{cL}{2}}{c} - \frac{2}{L}\int_0^{L/2} 2z'\cos cz'\,dz'$$

$$= 2\,\frac{\sin\frac{cL}{2}}{c} - \frac{4}{L}\left[\frac{\cos cz' + cz'\sin cz'}{c^2}\right]_0^{\frac{L}{2}}$$

$$= 2\,\frac{\sin\frac{cL}{2}}{c} - \frac{4}{L}\,\frac{\cos\frac{cL}{2} + \frac{cL}{2}\sin\frac{cL}{2} - 1}{c^2} = \frac{4}{L}\,\frac{1 - \cos\frac{cL}{2}}{c^2}$$

$$= L\,\frac{1 - \cos\frac{cL}{2}}{\left(\frac{cL}{2}\right)^2} = L\,\frac{2\sin^2\frac{cL}{4}}{\left(\frac{cL}{2}\right)^2} = \frac{L}{2}\,\frac{\sin^2\frac{cL}{4}}{\left(\frac{cL}{4}\right)^2} = \frac{L}{2}\,\frac{\sin^2\frac{u}{2}}{\left(\frac{u}{2}\right)^2}$$

$$f(u) = \left[\frac{\sin\frac{u}{2}}{\frac{u}{2}}\right]^2 \qquad u = c\frac{L}{2} = (\beta\cos\theta + \beta_0)\frac{L}{2}$$

(b) From the fact that $HP = 1.28\frac{\lambda}{L}$ (broadside)

$$\theta_{HP_R} = \frac{\pi}{2} - \frac{HP}{2} = \frac{\pi}{2} - 0.64\frac{\lambda}{L}$$

$$u_{HP_R} = \beta\cos\theta_{HP_R}\,\frac{L}{2} = \frac{\beta L}{2}\cos\left(\frac{\pi}{2} - 0.64\frac{\lambda}{L}\right)$$

$$= \frac{\beta L}{2}\sin 0.64\frac{\lambda}{L} \approx \frac{\beta L}{2}\left(0.64\frac{\lambda}{L}\right) = \frac{2\pi}{\lambda}\frac{L}{2}\left(0.64\frac{\lambda}{L}\right) = 0.64\pi$$

$$f(u_{HP_R}) = \left[\frac{\sin(0.64\pi/2)}{0.64\pi/2}\right]^2 = 0.705 \checkmark$$

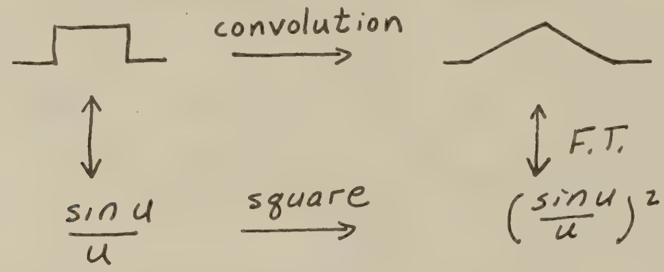
The position of $u/2$ for the first side lobe peak will be the same as u for a uniform line source. So $\quad (u/2)_{SLL} = \pm 1.43\pi$

4.2-5 (con't)

Then $f(u_{SLL}) = \left[\frac{\sin(1.43\pi)}{1.43\pi}\right]^2 = 0.0472 = -26.5\,dB$ ✓

4.2-6

The triangular current distribution is the convolution of two uniform line sources; i.e., a triangle is the convolution of two pulses.



4.2-7 Dipole with $L \ll \lambda$

(a) From (4-1)

$$E_\theta = j\omega\mu \frac{e^{-j\beta r}}{4\pi r} \sin\theta \underbrace{\int_{-L/2}^{L/2} I(z') e^{j\beta z'\cos\theta} dz'}_{}$$

$$= \frac{I_o L}{2}\left[\frac{\sin u/2}{u/2}\right]^2$$ from Prob. 4.2-5 with a maximum current of I_o instead of unity. And $u = \frac{\beta L}{2}\cos\theta$.

$$E_\theta = j\omega\mu \frac{e^{-j\beta r}}{4\pi r} \sin\theta \, \frac{I_o L}{2}\left[\frac{\sin(\frac{\beta L}{4}\cos\theta)}{\frac{\beta L}{4}\cos\theta}\right]^2$$

(b) For $L \ll \lambda$ $\quad \frac{\beta L}{4}\cos\theta = \frac{\pi L}{\lambda}\cos\theta \ll 1$, so $\left[\frac{\sin(\frac{\beta L}{4}\cos\theta)}{\frac{\beta L}{4}\cos\theta}\right]^2 \approx 1$

and

$$E_\theta \approx j\omega\mu \frac{I_o L}{2} \frac{e^{-j\beta r}}{4\pi r} \sin\theta$$

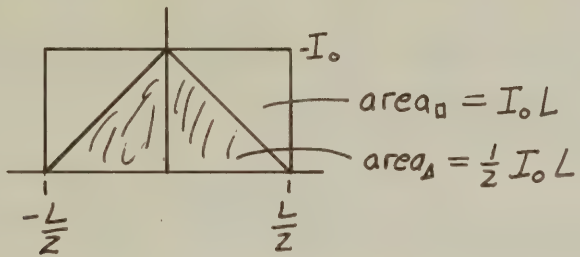
(c) From (1-71) for an ideal dipole

$$E_\theta = j\omega\mu \frac{e^{-j\beta r}}{4\pi r} I\,\Delta z \sin\theta$$

Comparing to short dipole result in (b) with $I = I_o$ and $\Delta z = L$ we see that <u>the triangular current (of short dipole) gives an expression for E_θ which is one-half that of a uniform current ideal dipole.</u>

4.2-7 (con't)

This is explained by noting that the triangle gives half the current-length as does the uniform current.



<u>4.2-8</u> $\quad I(z) = \cos^2\left(\frac{\pi z}{L}\right) \qquad |z| \le \frac{L}{2}$

$$f_{un} = \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos^2\left(\frac{\pi z'}{L}\right) e^{jcz'} dz' \text{ where } c = \beta\cos\theta + \beta_0 \text{ and } u = \frac{cL}{2}$$

$$= \frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} \left(1 + \cos\frac{2\pi z'}{L}\right) e^{jcz'} dz'$$

$$= \frac{1}{2}\left\{\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{jcz'} dz' + \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{e^{j\frac{2\pi}{L}z'} + e^{-j\frac{2\pi}{L}z'}}{2} e^{jcz'} dz'\right\}$$

$$= \frac{e^{jc\frac{L}{2}} - e^{-jc\frac{L}{2}}}{j2c} + \frac{1}{4}\left[\frac{e^{j(c+\frac{2\pi}{L})\frac{L}{2}} - e^{-j(c+\frac{2\pi}{L})\frac{L}{2}}}{j(c+\frac{2\pi}{L})} + \frac{e^{j(c-\frac{2\pi}{L})\frac{L}{2}} - e^{-j(c-\frac{2\pi}{L})\frac{L}{2}}}{j(c-\frac{2\pi}{L})}\right]$$

$$= \frac{L}{2}\frac{\sin\frac{cL}{2}}{\frac{cL}{2}} + \frac{1}{4}\left[\frac{2\sin\left[\frac{cL}{2}+\pi\right]}{c+\frac{2\pi}{L}} + \frac{2\sin\left[c\frac{L}{2}-\pi\right]}{c-\frac{2\pi}{L}}\right]$$

$$= \frac{L}{2}\frac{\sin u}{u} - \frac{1}{2}\frac{(c-\frac{2\pi}{L})\sin u + (c+\frac{2\pi}{L})\sin u}{c^2 - (\frac{2\pi}{L})^2}$$

$$= \frac{L}{2}\frac{\sin u}{u} - \frac{1}{2}\frac{2c\sin u}{c^2 - (\frac{2\pi}{L})^2} = \frac{L}{2}\frac{\sin u}{u}\left\{1 - \frac{2}{L}\frac{cu}{c^2\left[1 - (\frac{2}{cL}\pi)^2\right]}\right\}$$

$$= \frac{L}{2}\frac{\sin u}{u}\left\{1 - \frac{u}{\frac{cL}{2}\left[1-(\frac{\pi}{u})^2\right]}\right\} = \frac{L}{2}\frac{\sin u}{u}\frac{1-(\frac{\pi}{u})^2 - 1}{1-(\frac{\pi}{u})^2}$$

$$= \frac{L}{2}\frac{\sin u}{u}\frac{-(\frac{\pi}{u})^2}{1-(\frac{\pi}{u})^2} = \frac{L}{2}\frac{\sin u}{u}\frac{1}{1-(\frac{u}{\pi})^2}$$

$$\therefore f(u) = \frac{1}{1-(\frac{u}{\pi})^2}\frac{\sin u}{u} \text{ as in Table 4-2b}$$

4.2-8 (con't)

For broadside operation $HP = 1.44\frac{\lambda}{L}$

So $\theta_{HP_R} = \frac{\pi}{2} - \frac{HP}{2} = \frac{\pi}{2} - 0.72\frac{\lambda}{L}$

Then $u_{HP_R} = \frac{\beta L}{2}\cos\theta_{HP_R} = \frac{\pi L}{\lambda}\cos\left(\frac{\pi}{2} - 0.72\frac{\lambda}{L}\right) = \frac{\pi L}{\lambda}\sin\left(0.72\frac{\lambda}{L}\right)$

$$\approx \frac{\pi L}{\lambda} 0.72\frac{\lambda}{L} = 0.72\pi$$

And

$$f(u=u_{HP_R}) = \frac{1}{1-\left(\frac{0.72\pi}{\pi}\right)^2}\ \frac{\sin(0.72\pi)}{0.72\pi} = 0.7073 \ \checkmark$$

4.2-9

Cosine on a -10dB pedestal, $L = 20$ m, $\lambda = 1.5$ m @ 200 MHz

(a) From Table 4-2c

$$HP = 1.03\frac{\lambda}{L} = 1.03\frac{1.5}{20} = 0.07725 \text{ rad} = \boxed{4.43^\circ}$$

(b)

$$D = \frac{D}{D_u}D_u = (0.92)\,2\frac{L}{\lambda} = (0.92)(2)\frac{20}{1.5} = 24.53 = \boxed{13.9 \text{ dB}}$$

4.2-10

Cosine on a pedestal: $I(z) = C + (1-C)\cos\frac{\pi z}{L}$

$$f_{un} = \underbrace{C\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{j\beta z'\cos\theta}\,dz'}_{\text{see (4-4)}} + \underbrace{(1-C)\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\frac{\pi z'}{L} e^{j\beta z'\cos\theta}\,dz'}_{\text{see (4-25)}}$$

$$= CL\frac{\sin u}{u} + (1-C)\frac{2L}{\pi}\frac{\cos u}{1-\left(\frac{2}{\pi}u\right)^2}$$

To normalize

$$f_{un}(\max) = f_{un}(u=0) = CL + (1-C)\frac{2L}{\pi}$$

So

$$f(u) = \frac{f_{un}(u)}{f_{un}(\max)} = \frac{C\frac{\sin u}{u} + (1-C)\frac{2}{\pi}\frac{\cos u}{1-\left(\frac{2}{\pi}u\right)^2}}{C + (1-C)\frac{2}{\pi}}$$

which agrees with result in Table 4-2c.

4.2-11

$$D = \frac{2}{\lambda}\frac{\left|\int_{-L/2}^{L/2} I(z)\,dz\right|^2}{\int_{-L/2}^{L/2}|I(z)|^2\,dz} \quad \text{for a line source}$$

(a) ULS: $I(z) = 1 \quad |z| \le L/2$

$$\left|\int_{-L/2}^{L/2} 1\,dz\right|^2 = L^2 \ ; \ \int_{-L/2}^{L/2}|1|^2\,dz = L : D = \frac{2}{\lambda}\frac{L^2}{L} = \boxed{2\frac{L}{\lambda}}$$

4.2-11 (con't)

(b) $I(z) = C + (1-C)\cos\frac{\pi z}{L}$

$$\int_{-L/2}^{L/2} I(z)\,dz = C\int_{-\frac{L}{2}}^{\frac{L}{2}} dz + (1-C)\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\frac{\pi z}{L}\,dz$$

$$= CL + (1-C)\frac{\sin\frac{\pi}{2} - \sin(-\frac{\pi}{2})}{\pi/L} = CL + (1-C)\frac{2L}{\pi}$$

$$\int_{-L/2}^{L/2} |I(z)|^2 dz = \int_{-\frac{L}{2}}^{\frac{L}{2}}\left[C^2 + 2C(1-C)\cos\frac{\pi z}{L} + (1-C)^2\cos^2\frac{\pi z}{L}\right]dz$$

$$= C^2 L + 2C(1-C)\frac{2L}{\pi} + (1-C)^2\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1}{2}\left(1+\cos\frac{2\pi z}{L}\right)dz$$

$$= C^2 L + 2C(1-C)\frac{2L}{\pi} + (1-C)^2\frac{L}{2}$$

Thus

$$D = \frac{2}{\lambda}\frac{\left[CL + (1-C)\frac{2L}{\pi}\right]^2}{C^2 L + \frac{4}{\pi}C(1-C)L + (1-C)^2\frac{L}{2}}$$

$$\boxed{= \frac{2L}{\lambda}\frac{\left[C + (1-C)\frac{2}{\pi}\right]^2}{C^2 + \frac{4}{\pi}C(1-C) + \frac{1}{2}(1-C)^2}}$$

Evaluating this for various pedestal heights:

C	D/D_u	
1	1	Uniform line source
0.3162	0.927	−10 dB pedestal
0.1778	0.885	−15 dB pedestal
0	0.811	Cosine taper (no pedestal)

CHAPTER 5

5.1-1

From (5-3)

$$f_{un}=\int_{-\frac{L}{2}}^{0} I_m \sin\left[\beta\left(\tfrac{L}{2}+z'\right)\right]e^{j\beta z'\cos\theta}dz'+\int_{0}^{L/2} I_m \sin\left[\beta\left(\tfrac{L}{2}-z'\right)\right]e^{j\beta z'\cos\theta}dz'$$

$$=I_m\left\{\frac{e^{j\beta z'\cos\theta}}{\beta^2+(j\beta\cos\theta)^2}\left[j\beta\cos\theta\sin\left[\beta\left(\tfrac{L}{2}+z'\right)\right]-\beta\cos\left[\beta\left(\tfrac{L}{2}+z'\right)\right]\right\}_{-L/2}^{0}\right.$$

$$+I_m\left\{\frac{e^{j\beta z'\cos\theta}}{\beta^2+(j\beta\cos\theta)^2}\left[j\beta\cos\theta\sin\left[\beta\left(\tfrac{L}{2}-z'\right)\right]+\beta\cos\left[\beta\left(\tfrac{L}{2}-z'\right)\right]\right\}_{0}^{L/2}\right.$$

using (F-11)

$$=\frac{I_m}{\beta^2(1-\cos^2\theta)}\,\beta\left\{j\cos\theta\sin\frac{\beta L}{2}-\cos\frac{\beta L}{2}+e^{-j\frac{\beta L}{2}\cos\theta}\right.$$

$$\left.+e^{j\frac{\beta L}{2}\cos\theta}-j\cos\theta\sin\frac{\beta L}{2}-\cos\frac{\beta L}{2}\right\}$$

$$=\frac{I_m}{\beta\sin^2\theta}\left[2\cos\left(\frac{\beta L}{2}\cos\theta\right)-2\cos\frac{\beta L}{2}\right]$$ which is (5-4).

5.1-2

From (5-6)

$$E_\theta\sim\frac{\cos\left(\frac{\beta L}{2}\cos\theta\right)-\cos\left(\frac{\beta L}{2}\right)}{\sin\theta}$$

And from (B-18) $\cos\alpha\approx 1-\frac{\alpha^2}{2}$ for $\alpha\ll 1$

Then for $L\ll\lambda$

$$\cos\left(\frac{\beta L}{2}\cos\theta\right)-\cos\left(\frac{\beta L}{2}\right)\approx 1-\frac{1}{2}\left(\frac{\beta L}{2}\cos\theta\right)^2-1+\frac{1}{2}\left(\frac{\beta L}{2}\right)^2$$

$$=\frac{1}{2}\left(\frac{\beta L}{2}\right)^2(1-\cos^2\theta)=\frac{1}{2}\left(\frac{\beta L}{2}\right)^2\sin^2\theta$$ note: $\frac{\beta L}{2}=\frac{\pi L}{\lambda}\ll 1$

So

$$E_\theta\sim\frac{1}{2}\left(\frac{\beta L}{2}\right)^2\frac{\sin^2\theta}{\sin\theta}\sim\sin\theta \qquad \therefore \boxed{F(\theta)=\sin\theta}$$

5.1-3

(a) $F_a(\theta)=g_a(\theta)\,f_a(\theta)=\dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\cdot\dfrac{\sin\psi}{2\sin\psi/2}$ $=(\;)\dfrac{2\sin\psi/2\cos\psi/2}{2\sin\psi/2}$

$\psi=\beta d\cos\theta+\alpha=\pi\cos\theta \qquad d=\lambda/2$ $\qquad =(\;)\cos\psi/2$

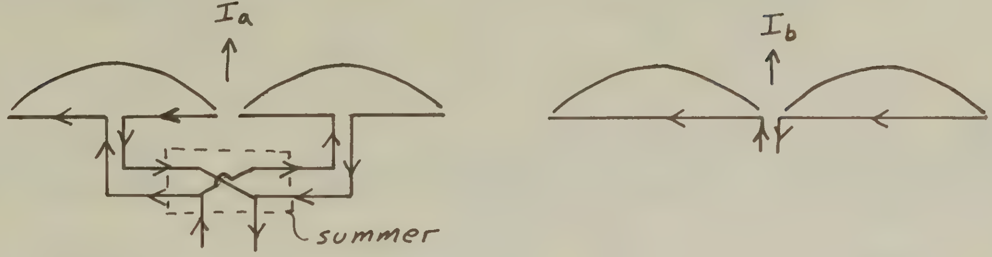
$F_a(\theta)=\dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\;\dfrac{\sin(\pi\cos\theta)}{2\sin\left(\frac{\pi}{2}\cos\theta\right)}$ $\qquad =(\;)\cos\left(\frac{\pi}{2}\cos\theta\right)$

$$=\frac{\cos^2(\pi/2\cos\theta)}{\sin\theta}$$

5.1-3 (con't)

(b) $$F_b(\theta) = \frac{\cos(\pi\cos\theta) + 1}{2\sin\theta} \qquad (5\text{-}8)$$

(c)



The current distributions are the same, so the patterns must be also!

Note: From (B-10)

$$\sin(\pi\cos\theta) = 2\sin\left(\tfrac{\pi}{2}\cos\theta\right)\cos\left(\tfrac{\pi}{2}\cos\theta\right)$$

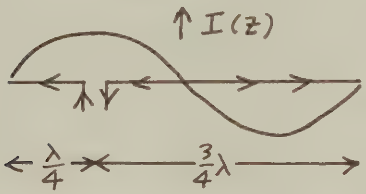
so $$\frac{\sin(\pi\cos\theta)}{2\sin\left(\frac{\pi}{2}\cos\theta\right)} = \cos\left(\tfrac{\pi}{2}\cos\theta\right)$$

and

$$F_a = \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\cos\left(\tfrac{\pi}{2}\cos\theta\right) = \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} = \frac{\frac{1}{2}\left(1+\cos\left(2\frac{\pi}{2}\cos\theta\right)\right)}{\sin\theta}$$

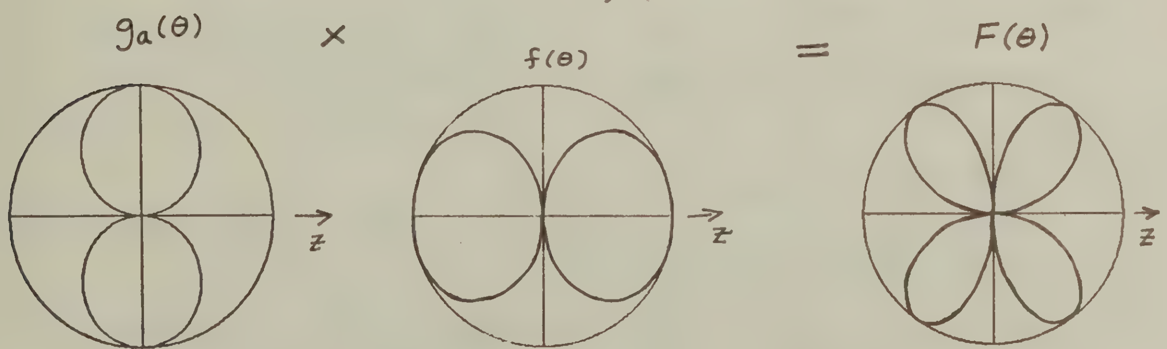
$$= \frac{\cos(\pi\cos\theta)+1}{2\sin\theta} = F_b(\theta) \quad \checkmark$$

5.1-4

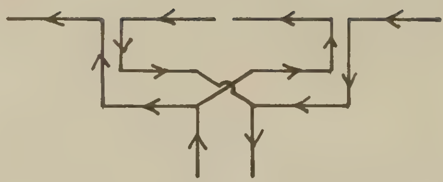


Imagine that this current is generated by two half-wave dipoles, a half wavelength apart and fed 180° out of phase.

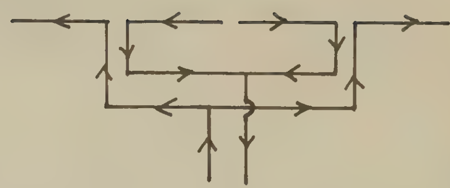
Then $$F(\theta) = g_a(\theta)\,f(\theta) = \underset{(5\text{-}7)}{\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}} \cdot \underset{(3\text{-}8)}{\sin\left(\tfrac{\pi}{2}\cos\theta\right)}$$



5.1-5 (a)



(b)



5.1-6

From (4-27) $\quad f(u) = \dfrac{\cos u}{1-(\frac{2}{\pi}u)^2}$ where $u = \frac{\beta L}{2}(\cos\theta - \cos\theta_0)$

For a half-wave dipole, the current distribution has this cosine shape and $L = \lambda/2$ and $\theta_0 = 90°$; $u = \frac{\pi}{2}\cos\theta$.

And

$$f(\theta) = \frac{\cos(\frac{\pi}{2}\cos\theta)}{1-(\frac{2}{\pi}\frac{\pi}{2}\cos\theta)^2} = \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$

The complete pattern is

$$F(\theta) = g(\theta)\,f(\theta) = \sin\theta\,\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} = \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}$$

which is (5-7) as required.

5.1-7

By trial and error $\dfrac{\cos(\frac{3}{2}\pi\cos\theta)}{\sin\theta}$ from (5-9) for a $\frac{3}{2}\lambda$ dipole is maximum at $\underline{\theta_0 = 42.6°\text{ and }137.4°}$ where its value is 1.399. So the normalization constant is $K = 1/1.399 = \underline{0.7148}$.

5.1-8

$\lambda = 1.695$ m @ 177 MHz

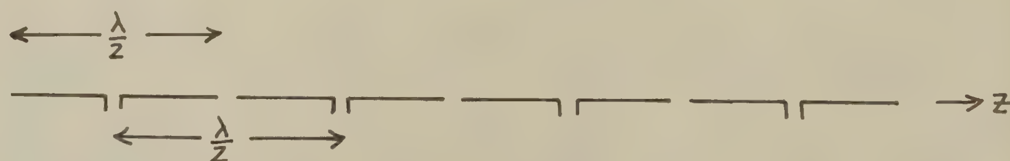
To determine the shortening required:

$$\frac{\text{Length}}{\text{Diameter}} = \frac{L}{2a} \approx \frac{\lambda/2}{(\frac{1}{2}'')} = \frac{169.5\text{ cm}/2}{2.54\text{ cm}/2} = 66.7$$

which is close to an $L/2a$ of 50 in Table 5-2, so

$L = 0.475\lambda = 0.475(169.5) = \boxed{80.5\text{ cm}}$

5.1-9



(a) Element pattern $= g_a(\theta) = \dfrac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}$ $\qquad$ (5-7)

5.1-9 (con't)

Array factor $= f(\psi) = \dfrac{\sin N\frac{\psi}{2}}{N\sin\frac{\psi}{2}}$ $\quad N=4 \quad d=\lambda/2 \quad \alpha=0$

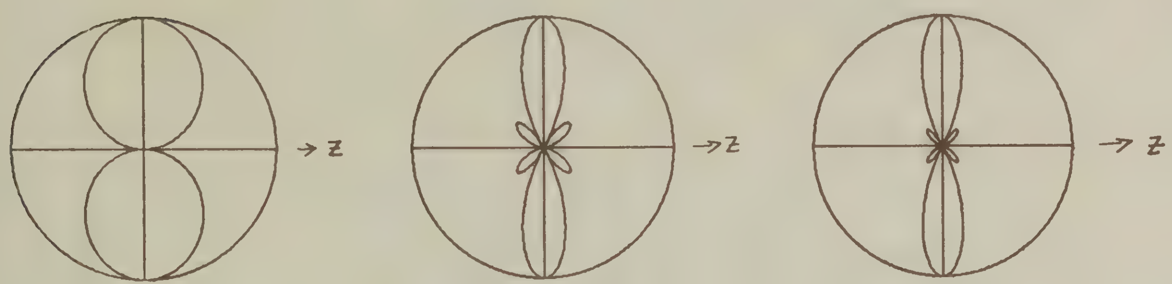
$$\psi = \beta d\cos\theta + \alpha = \frac{2\pi}{\lambda}\frac{\lambda}{2}\cos\theta + 0 = \pi\cos\theta$$

$$f(\theta) = \frac{\sin(2\pi\cos\theta)}{4\sin(\frac{\pi}{2}\cos\theta)}$$

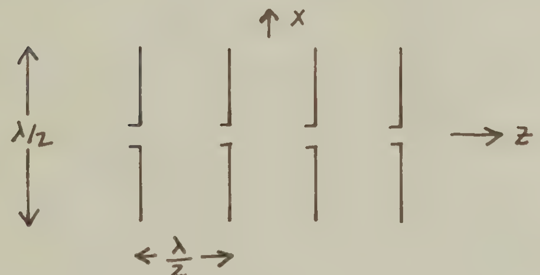
Total pattern $F(\theta) = g_a(\theta) f(\theta) = \dfrac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \dfrac{\sin(2\pi\cos\theta)}{4\sin(\frac{\pi}{2}\cos\theta)}$

(b)

$g_a(\theta) \quad \times \quad f(\theta) \quad = \quad F(\theta)$



5.1-10



(a) The array factor is

$$f(\psi) = \frac{\sin N\frac{\psi}{2}}{N\sin\frac{\psi}{2}} \qquad \psi = \beta d\cos\theta + \alpha = \pi\cos\theta - \pi$$

since $d = \lambda/2$ and $\alpha = -\beta d\cos\theta_0 = -\pi\cos 0 = -\pi$

$$f(\theta) = \frac{\sin[2\pi(\cos\theta - 1)]}{2\sin[\frac{\pi}{2}(\cos\theta - 1)]}$$

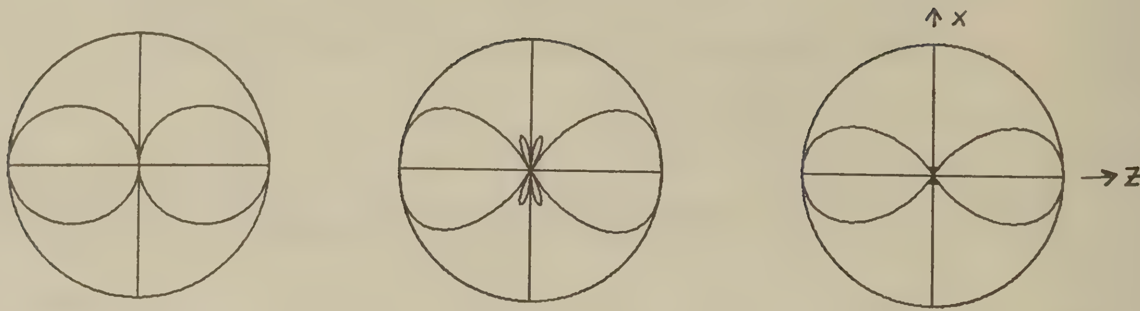
The total pattern is

$$F(\theta) = g_a(\theta,\phi) f(\theta) = \underbrace{\frac{\cos(\frac{\pi}{2}\sin\theta\cos\phi)}{\sqrt{1-\sin^2\theta\cos^2\phi}}}_{\text{from (3-69)}} \frac{\sin(2\pi(\cos\theta - 1))}{2\sin(\frac{\pi}{2}(\cos\theta - 1))}$$

5.1-10 (con't)

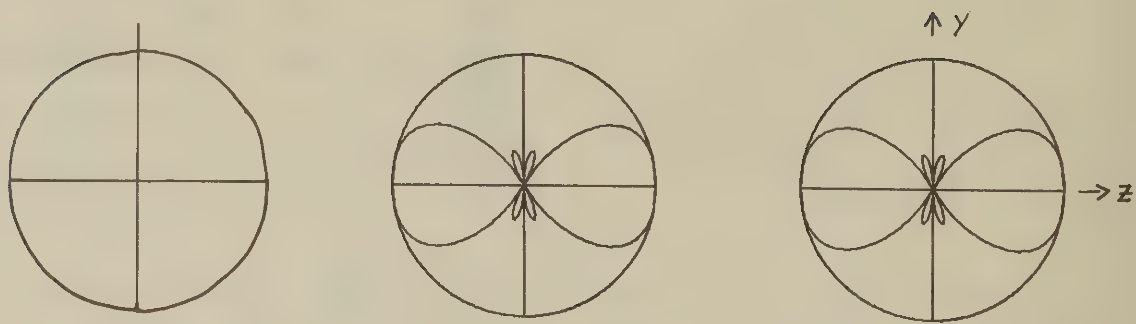
(b) In xz-plane

$g_a \quad \times \quad f \quad = \quad F$



In yz-plane

$g_a \quad \times \quad f \quad = \quad F$

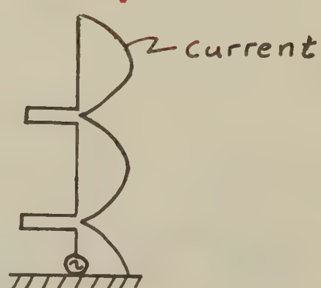


In xy-plane $F=0$ since f does.

5.1-11

(a)

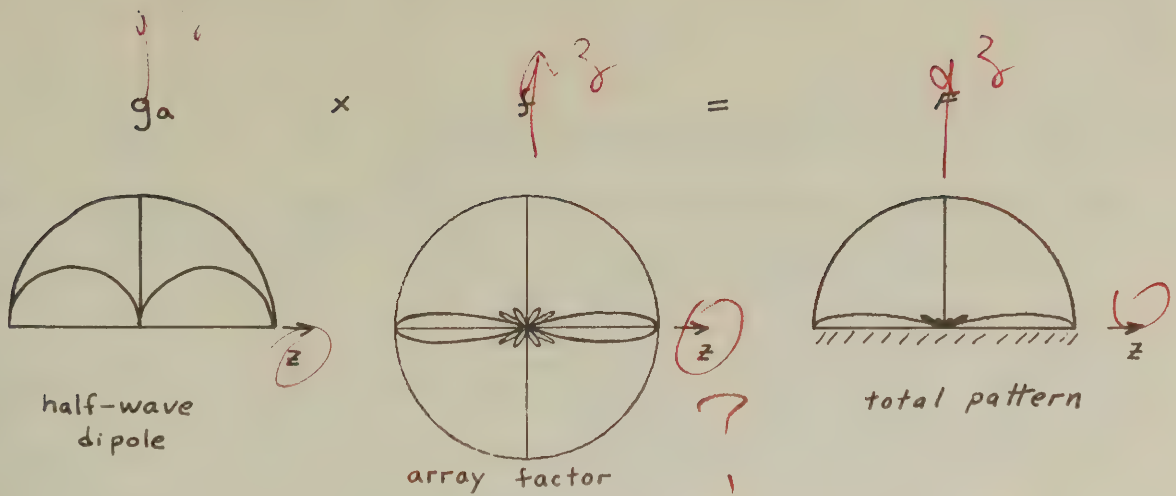
The quarter-wave stubs act as 180° phase shifters.



(b)

The antenna with its ground plane image forms an array of half-wave dipoles with equal amplitudes and

$N=6$, $d=\lambda/2$, $\alpha=0$



5.1-12

(a) From (5-11)

$$P_r = \frac{\eta}{4\pi} I_m^2 \int_0^1 \left\{ \frac{[\cos\frac{\beta L}{2}\tau - \cos\frac{\beta L}{2}]^2}{1+\tau} + \frac{[\cos\frac{\beta L}{2}\tau - \cos\frac{\beta L}{2}]^2}{1-\tau} \right\} d\tau$$

Let $v = 1+\tau \qquad w = 1-\tau$

Then

$$P_r = \frac{\eta}{4\pi} I_m^2 \left\{ \int_1^2 \frac{[\cos\frac{\beta L}{2}(1-v) - \cos\frac{\beta L}{2}]^2}{v} (+dv) + \int_1^0 \frac{[\cos\frac{\beta L}{2}(1-w) - \cos\frac{\beta L}{2}]^2}{w} (-dw) \right\}$$

And after several manipulations

$$[\cos\tfrac{\beta L}{2}(1-v) - \cos\tfrac{\beta L}{2}]^2 = 1 - \cos\tfrac{\beta L}{2}v + \cos\beta L(1-\cos\tfrac{\beta L}{2}v)$$
$$-\tfrac{1}{2}\cos\beta L(1-\cos\beta L v) + \sin\beta L(\tfrac{1}{2}\sin\beta L v - \sin\tfrac{\beta L}{2}v)$$

Since the integrands are identical, we can combine the two integrals giving

$$\mathcal{I} = \int_0^2 \frac{1-\cos\frac{\beta L}{2}v}{v} dv + \cos\beta L \int_0^2 \frac{1-\cos\frac{\beta L}{2}v}{v} dv$$
$$-\tfrac{1}{2}\cos\beta L\int_0^2 \frac{1-\cos\beta L v}{v} dv + \tfrac{1}{2}\sin\beta L \int_0^2 \frac{\sin\beta L v}{v} dv - \sin\beta L \int_0^2 \frac{\sin\frac{\beta L}{2}v}{v} dv$$

Now let $\frac{\beta L}{2}v = x$ or $\beta L v = y$

Then

$$\mathcal{I} = \int_0^{\beta L} \frac{1-\cos x}{x} dx + \cos\beta L \int_0^{\beta L} \frac{1-\cos x}{x} dx - \tfrac{1}{2}\cos\beta L \int_0^{2\beta L} \frac{1-\cos y}{y} dy$$
$$+\tfrac{1}{2}\sin\beta L \int_0^{2\beta L} \frac{\sin y}{y} dy - \sin\beta L \int_0^{\beta L} \frac{\sin x}{x} dx$$

Using (F-13) and (F-14)

$$\mathcal{I} = 0.5772 + \ln(\beta L) - Ci(\beta L) + \cos\beta L \{0.5772 + \ln(\beta L)$$
$$- Ci(\beta L) - \tfrac{1}{2}[0.5772 + \ln(\beta L) - Ci(2\beta L)]\}$$

5.1-12 (con't)

$$+ \sin\beta L\left[\tfrac{1}{2}Si(2\beta L) - Si(\beta L)\right]$$

$$\mathscr{I} = 0.5772 + \ln(\beta L) - Ci(\beta L) + \tfrac{1}{2}\sin\beta L\left[Si(2\beta L) - 2\,Si(\beta L)\right]$$
$$+ \tfrac{1}{2}\cos\beta L\left[0.5772 + \underbrace{2\ln(\beta L) - \ln(2\beta L)}_{=\ln\frac{\beta L}{2}} - 2\,Ci(\beta L) + Ci(2\beta L)\right]$$

Thus

$$P_r = \frac{\eta}{4\pi} I_m^2 \Big\{ 0.5772 + \ln(\beta L) - Ci(\beta L) + \tfrac{1}{2}\sin\beta L\left[Si(2\beta L) - 2Si(\beta L)\right]$$
$$+ \tfrac{1}{2}\cos\beta L\left[0.5772 + \ln\left(\tfrac{\beta L}{2}\right) + Ci(2\beta L) - 2\,Ci(\beta L)\right]\Big\}$$

as claimed.

(b) The directivity is $D = \dfrac{4\pi U_m}{P_r}$

And

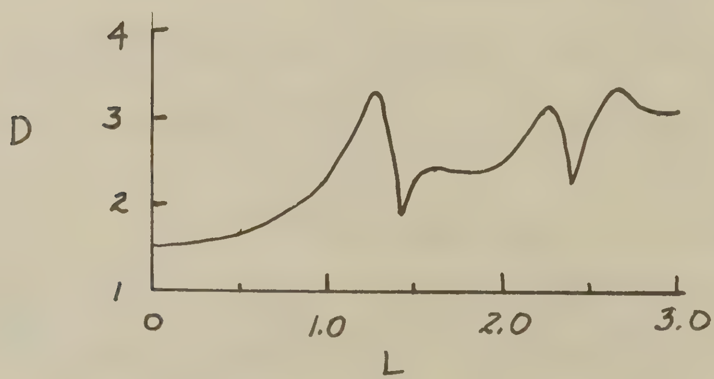
$$U_m = \frac{r^2}{2\eta}\left|E_\theta\right|^2_{max} = \frac{\eta}{8\pi^2} I_m^2 \left|\frac{\cos\left(\frac{\beta L}{2}\cos\theta\right) - \cos\left(\frac{\beta L}{2}\right)}{\sin\theta}\right|^2_{max} \quad \text{from (5-6)}$$

Using P_r from (a)

$$D = \frac{4\pi \frac{\eta}{8\pi^2} I_m^2\, A}{\frac{\eta}{4\pi} I_m^2\, B} = 2\,\frac{A}{B} \qquad \text{where}$$

$$A = \left|\frac{\cos\left(\frac{\beta L}{2}\cos\theta\right) - \cos\left(\frac{\beta L}{2}\right)}{\sin\theta}\right|^2_{max}$$

$$B = 0.5772 - \ln(\beta L) - Ci(\beta L) + \tfrac{1}{2}\sin(\beta L)\left[Si(2\beta L) - 2Si(\beta L)\right]$$
$$+ \tfrac{1}{2}\cos\beta L\left[0.5772 + \ln\left(\tfrac{\beta L}{2}\right) + Ci(2\beta L) - 2\,Ci(\beta L)\right]$$



5.1-13

For $a = 0.005\,m \quad L = 0.5\,m$

$$\frac{L}{2a} = \frac{0.5}{0.1} = 50 \Rightarrow L = 0.475\lambda \text{ at resonance}$$

When $L = 0.475\lambda$, $\lambda = L/0.475 = 0.5m/0.475 = 1.0526\ m$

$$f = c/\lambda = 3\times10^8/1.0526 = 2.85\times10^8 = \boxed{285\ MHz}$$

5.1-13 (con't)

For $a=0.0001$ $\quad \frac{L}{2a} = \frac{0.5}{0.0002} = 2500 \Rightarrow L = 0.49\lambda$ at resonance

Then $\quad \lambda = \frac{L}{0.49} = \frac{0.5}{0.49} = 1.0204\text{ m}$

And $\quad f = c/\lambda = 3\times10^8/1.0204 = 2.94\times10^8 = \boxed{294\text{ MHz}}$

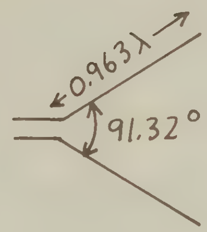
These values agree well with the VSWR=1 points in Fig. 5-7.

5.1-14 Optimum directivity vee

$D(\text{dB}) = 6 \Rightarrow D = 10^{6/10} = 3.98$

From (5-24)

$$\frac{h}{\lambda} = \frac{D-1.15}{2.94} = \frac{3.98-1.15}{2.94} = 0.963$$



Then from (5-23)

$$\gamma = 152\left(\frac{h}{\lambda}\right)^2 - 388\left(\frac{h}{\lambda}\right) + 324$$

$$= 152(0.963)^2 - 388(0.963) + 324 = \boxed{91.32^\circ}$$

5.1-15

$D = 2.41$

From (5-24)

$$\frac{h}{\lambda} = \frac{D-1.15}{2.94} = \frac{2.41-1.15}{2.94} = 0.43 \; ; \; L = 2h = 2(0.43\lambda) = 0.86\lambda$$

From (5-23)

$$\gamma = 152\left(\frac{h}{\lambda}\right)^2 - 388\left(\frac{h}{\lambda}\right) + 324 = \underline{185.63^\circ}$$

A straight wire dipole would have $\gamma = 180^\circ$, so this result is close.

5.2-1

(a) $\frac{\lambda}{2} = \frac{1}{2}\frac{3\times10^8}{f} = \frac{1.5\times10^8}{f(\text{MHz})\times10^6} = \frac{150}{f(\text{MHz})}$

For thin wire antenna from Table 5-2

$$L = 0.475\lambda = 0.95\left(\frac{\lambda}{2}\right) = \frac{0.95(150)}{f(\text{MHz})} = \frac{142.5}{f(\text{MHz})} \; ; \; \boxed{L(\text{cm}) = \frac{14250}{f(\text{MHz})}}$$

(b)

Channel	f(MHz)	L(cm)	Channel	f(MHz)	L(cm)
2	57	250	7	177	80.5
3	63	226.2	8	183	77.9
4	69	206.5	9	189	75.4
5	79	180.4	10	195	73.1
6	85	167.6	11	201	70.9
FM	100	142.5	12	207	68.8
			13	213	66.9

5.2-2

$Z_o = 300\,\Omega$ since $d = 12.5a$

From Figs. 5-5 and 5-6 for $L = 0.4\lambda$ $\quad Z_D = 35 - j170\,\Omega$

From (5-25)

$$Z_T = j Z_o \tan\frac{\beta L}{2} = j\,300 \tan\frac{2\pi}{\lambda}\frac{0.4\lambda}{2} = j923.3\,\Omega$$

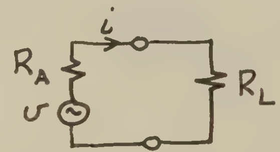
From (5-29)

$$Z_{in} = \frac{4 Z_T Z_D}{Z_T + 2 Z_D} = \frac{4(j923.3)(35-j170)}{j923.3 + 2(35-j170)} = \boxed{345.8 - j1034.9\,\Omega}$$

From Fig. 5-15

$$Z_{in}\,(L = 0.4\lambda) = 330 - j1000\,\Omega$$

5.3-1



$$i = \frac{v}{R_A + R_L} \quad ; v_L = iR_L = \frac{R_L}{R_A + R_L} v$$

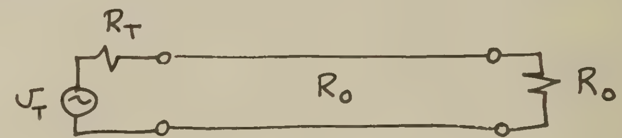
$$P_L = v_L i = \frac{R_L}{(R_A + R_L)^2} v^2$$

$$0 = \frac{\partial P_L}{\partial R_L} = \left[\frac{1}{(R_A + R_L)^2} + R_L(-2)(R_A + R_L)^{-3}\right] v^2$$

$$0 = (R_A + R_L) - 2R_L \quad \text{or} \quad 0 = R_A - R_L \quad \text{or} \quad \boxed{R_L = R_A}$$

5.3-2

(a)



$$i = \frac{v_T}{R_T + R_o}$$

Input power $= p_{in} = i^2 R_o = \left(\frac{v_T}{R_T + R_o}\right)^2 R_o$

Power dissipated in transmitter $= i^2 R_T = \left(\frac{v_T}{R_T + R_o}\right)^2 R_T$

Transmit efficiency $= e_t = \dfrac{P_{in}}{P_{total}}$

$$= \frac{\frac{v_T^2}{(R_T+R_o)^2} R_o}{\frac{v_T^2}{(R_T+R_o)^2} R_o + \frac{v_T^2}{(R_T+R_o)^2} R_T} = \frac{R_o}{R_o + R_T} = \boxed{\frac{1}{1 + R_T/R_o}}$$

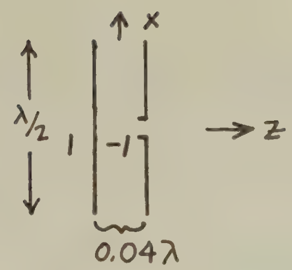
(b)

$R_T = R_o \quad e_t = \dfrac{1}{1+1} = \dfrac{1}{2} = \boxed{50\%}$

$R_T = 0.5 R_o \quad e_t = \dfrac{1}{1+0.5} = \dfrac{1}{1.5} = \boxed{66.7\%}$

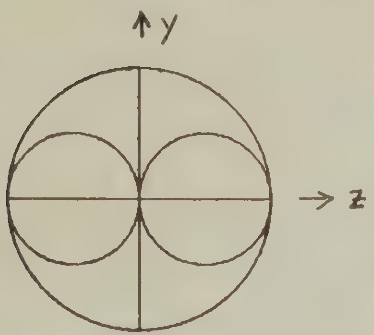
$R_T = 0.1 R_o \quad e_t = \dfrac{1}{1+0.1} = \dfrac{1}{1.1} = \boxed{91.1\%}$

5.4-1



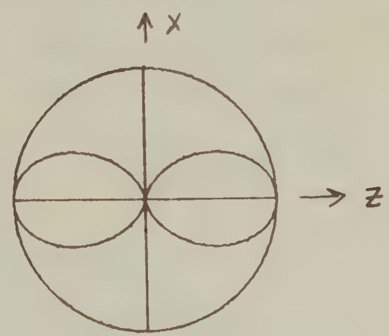
Using the ARRPAT program:

H-plane



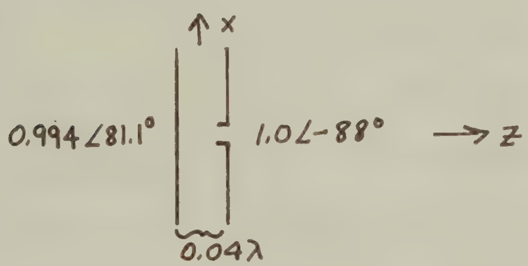
(a)

E-plane



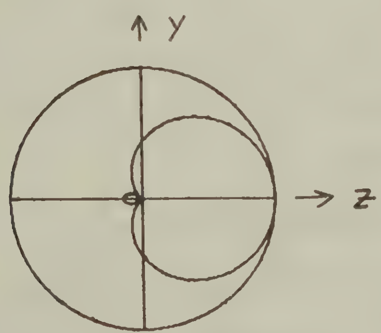
(b)

5.4-2

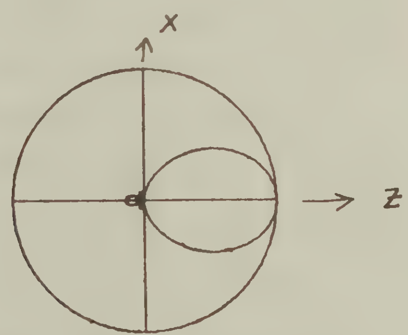


Using ARRPAT:

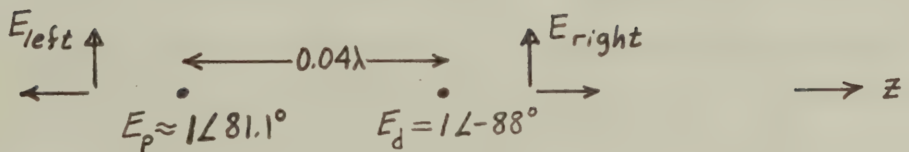
H-plane



E-plane



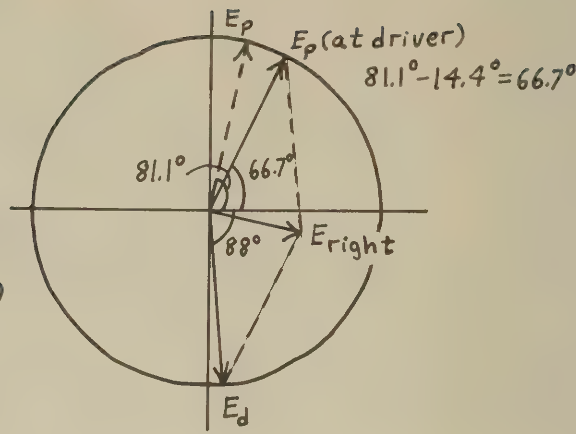
5.4-3



$\beta d = \frac{2\pi}{\lambda} 0.04\lambda = 0.08\pi = 14.4°$

5.4-3 (con't)

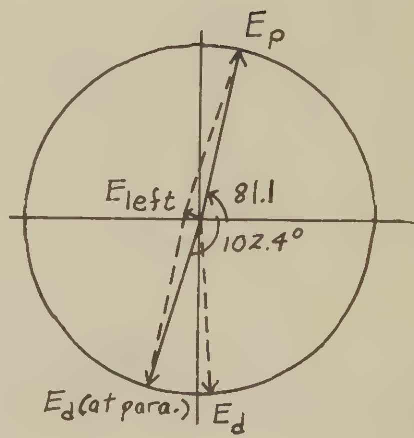
Right ($\theta = 0°$)



$$E_{right} = 1\angle{-88°} + 1\angle 66.7°$$
$$= 0.438\angle{-10.65°}$$

This is the sum of E_d and E_p(at driver) where E_p(at driver) is $1\angle 81.1°$ delayed by $14.4°$ due to βd phase shift giving $1\angle 66.7°$

Left ($\theta = 180°$)



$$E_{left} = E_p + E_d(\text{at parasite})$$
$$= 1\angle 81.1° + 1\angle{-88° - 14.4°}$$
$$= 1\angle 81.1° + 1\angle{-102.4°}$$
$$= 0.061\angle 180° - 10.65°$$
$$= 0.061\angle 169.35°$$

The front-to-back ratio is

$$\frac{0.438}{0.061} = 7.18 = \boxed{17.1\ dB}$$

The pattern in Prob. 5.4-2 (which is the same array except the parasite amplitude is 0.994 and here it is 1.0) has a front-to-back ratio of 17.0 dB.

<u>5.4-4</u> $\lambda = 3m = 300cm$ @ 100 MHz

From Table 5-4 for a three element Yagi:

Spacing $= S_R = S_D = 0.25\lambda = 0.25(300) = \boxed{75cm}$

Reflector length $= L_R = 0.479\lambda = 0.479(300) = \boxed{143.7cm}$

Driver length $= L = 0.453\lambda = 0.453(300) = \boxed{135.9\ cm}$

Director length $= L_D = 0.451\lambda = 0.451(300) = \boxed{135.3\ cm}$

5.4-5

For channel 13 $f = 213$ MHz (center frequency) $\Rightarrow \lambda = 140.8$ cm

Then From Table 5-4

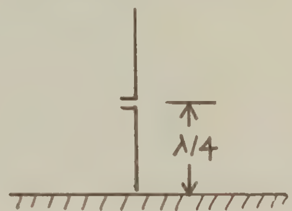
$$L_R = 0.477\lambda = \underline{67.2\text{ cm}} \qquad L = 0.454\lambda = \underline{63.9\text{ cm}}$$

$$L_D = 0.434\lambda = \underline{61.1\text{ cm}} \qquad S_R = S_D = 0.25\lambda = \underline{35.2\text{ cm}}$$

One reflector, one driver, five directors.

5.5-1

From (3-97) and (3-98)



$$Z_{1,in} = Z_{11} + \frac{I_2}{I_1} Z_{12}$$

Let #1 be the antenna and #2 be its image in the ground plane.

The ground plane presents an image of equal amplitude and the same phase; so $I_2 = I_1$

So

$$Z_{1,in} = Z_{11} + Z_{12}$$

From Fig. 3-27 for $d/\lambda = 0.5$ $\qquad Z_{12} = 22 + j10$

And for a half-wave (thin) dipole $\qquad Z_{11} \approx 70 + j0$

Thus $\quad Z_{1,in} = 70 + j0 + 22 + j0 = \boxed{92 + j10 \text{ ohms}}$

5.5-2

From Table 3-1 for collinear short dipoles

$$a_0 = \frac{2}{3} \qquad a_1 = \frac{2}{(m\beta d)^2} \qquad a_2 = \frac{-2}{m\beta d}$$

Borrowing the results of (2-22)

$$D_{\text{above ground plane}} = 2\, D_{\text{free space array}}$$

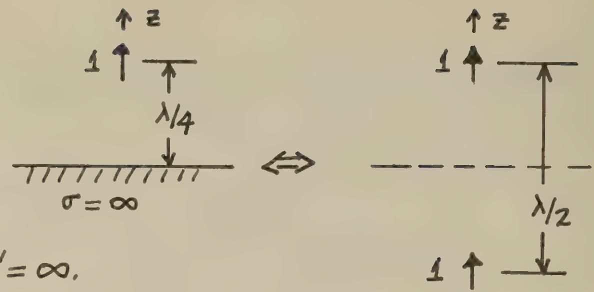
$$= \frac{2}{\frac{a_0}{N} + \frac{2}{N^2}\sum_{m=1}^{N-1} \frac{N-m}{m\beta d}(a_1 \sin m\beta d + a_2 \cos m\beta d)\cos m\alpha} \quad \text{using (3-83)}$$

$$= \frac{2}{\frac{2}{3}\frac{1}{2} + \frac{2}{4}\left[\frac{2-1}{\beta d}\left(\frac{2}{(\beta d)^2}\sin\beta d + \frac{-2}{\beta d}\cos\beta d\right)\right]} \quad \text{using Table 3-1, } N=2, \text{ and } \alpha = 0$$

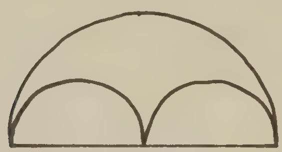
$$= 2\left[\frac{1}{3} - \frac{\cos(\beta 2h)}{(\beta 2h)^2} + \frac{\sin(\beta 2h)}{(\beta 2h)^3}\right]^{-1} \quad \text{where } d = 2h.$$

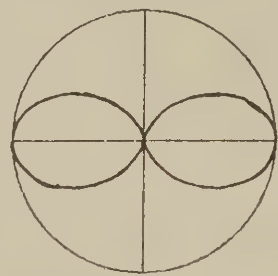
5.5-3

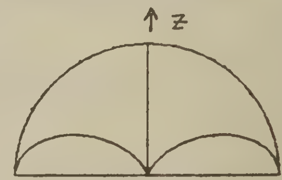
(a) The image current value of +1 is obtained from image theory in Sec. 2.3 or from (5-47) with $\Gamma_V = +1$ from (5-53) with $\epsilon_r' = \infty$.



E-plane element pattern × array factor (see Fig. 3-3) = Total E-plane pat.



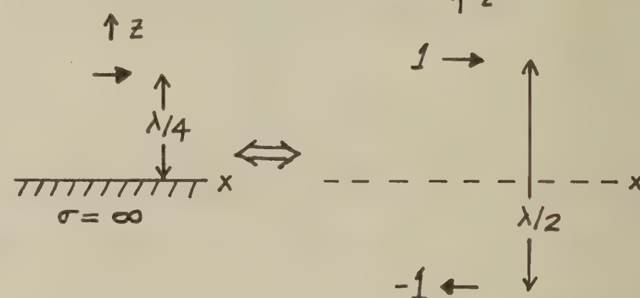




which comparés with Fig. 5-37.

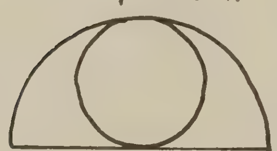
The H-plane (xy-plane) pattern is isotropic.

(b) The image current −1 is obtained from image theory in Sec. 2.3 or from (5-50) with $\Gamma_V = +1$ for $\epsilon_r' = \infty$ in (5-53) or from (5-51) with $\Gamma_H = -1$ for $\epsilon_r' = \infty$ in (5-52).

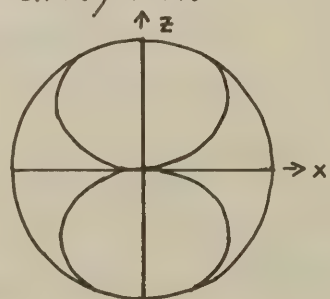


E-plane

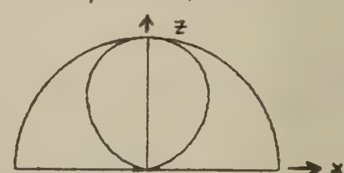
element pattern × array factor = Total E-plane pattern



(sin θ)

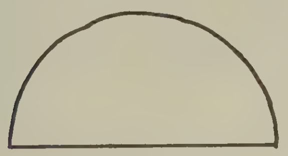


(Fig. 3-4)

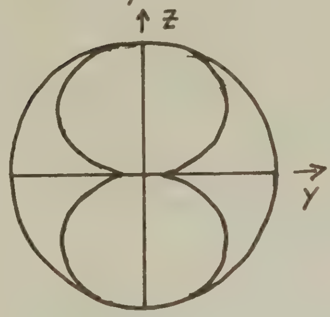


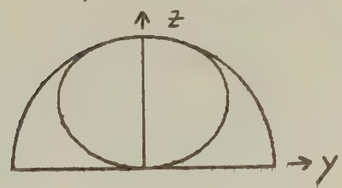
5.5-3 (con't)

H-plane element pattern X array factor = Total H-plane pattern



(isotropic)





which compares to Fig. 5-38a.

5.5-4

The program ARRPAT is used to give

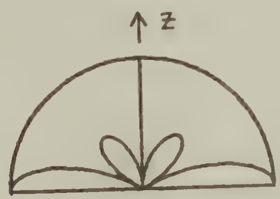
(a)

THE ELEMENTS ARE COLLINEAR SHORT DIPOLES ALONG THE Z-AXIS

ELEMENT LOCATIONS, CURRENTS, AND PHASES

I	X(I)	Y(I)	Z(I)	A(I)	ALPHA(I)
1	0.0	0.0	0.0	1.0000	0.0
2	0.0	0.0	1.0000	1.0000	0.0

E-plane & H-plane pattern:



which compares to Fig. 5-37b.

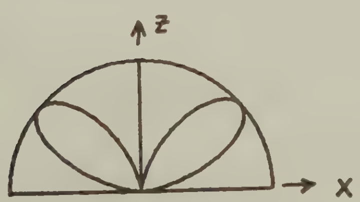
(b)

THE ELEMENTS ARE SHORT DIPOLES PARALLEL TO X-AXIS

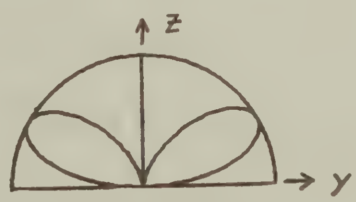
ELEMENT LOCATIONS, CURRENTS, AND PHASES

I	X(I)	Y(I)	Z(I)	A(I)	ALPHA(I)
1	0.0	0.0	0.0	1.0000	0.0
2	0.0	0.0	1.0000	1.0000	179.9999

E-plane pattern



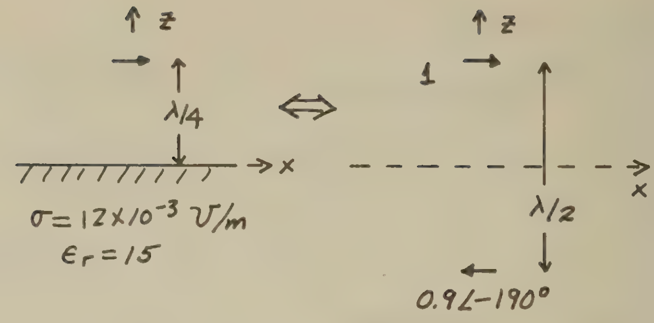
H-plane pattern

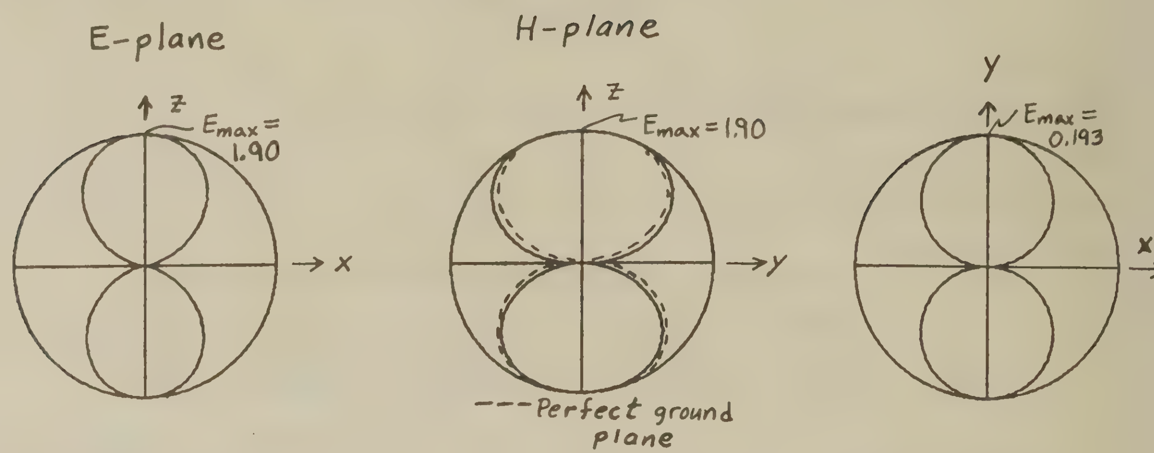


which compares to Fig. 5-38b

5.5-5

(a) The program ARRPAT was used to obtain the patterns. The lower pattern halves have been included to show unsymmetrical nature.





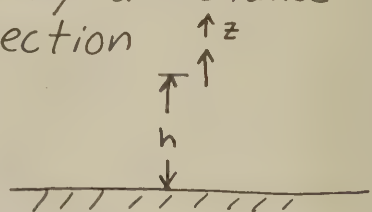
(b) The H-plane pattern is very close that for the perfect ground plane case of Prob. 5.5-3b (shown dashed above) except that radiation is not zero along the ground plane (xy-plane) but is $(0.193/1.90 = -19.9\,dB)$ 19.9 dB below the main beam maximum.

5.5-6

(a) From (5-49)

$$E_\theta = j\omega\mu \frac{IL}{4\pi}\frac{e^{-j\beta r}}{r}\sin\theta\left(e^{j\beta h\cos\theta} + \Gamma_v e^{-j\beta h\cos\theta}\right)$$

for a short dipole oriented vertically a distance h above a ground plane with reflection coefficient Γ_v.



So the unnormalized pattern is

$$F_u(\theta) = \sin\theta\left(e^{j\beta h\cos\theta} + \Gamma_v e^{-j\beta h\cos\theta}\right)$$

$$= \sin\theta\left[e^{j\pi\cos\theta} + A\, e^{j(-\pi\cos\theta + \delta)}\right]$$

for $h = \frac{\lambda}{2}$ and $\Gamma_v = Ae^{j\delta}$

5.5-6 (con't)

And

$$|F_u(\theta)|^2 = F_u(\theta)\, F_u^*(\theta) = \sin^2\theta\left[e^{j\pi\cos\theta} + Ae^{j(-\pi\cos\theta+\delta)}\right] \cdot \left[e^{-j\pi\cos\theta} + Ae^{-j(-\pi\cos\theta+\delta)}\right]$$

$$= \sin^2\theta\left[1 + A^2 + Ae^{j2\pi\cos\theta}e^{-j\delta} + Ae^{-j2\pi\cos\theta}e^{j\delta}\right]$$

$$= \sin^2\theta\left[1 + A^2 + 2A\cos(2\pi\cos\theta + \delta)\right]$$

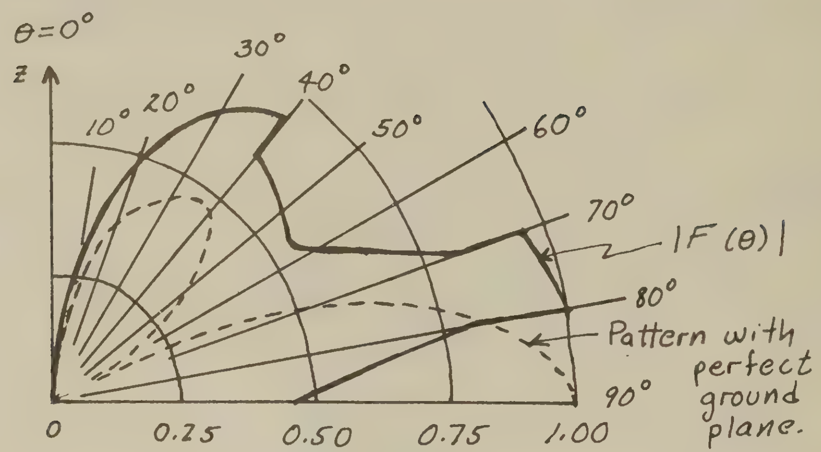
Tabulating

θ	Γ_v: A	Γ_v: δ	$\lvert F_u(\theta)\rvert^2 = \sin^2\theta[1+A^2+2A\cos(2\pi\cos\theta+\delta)]$	$\lvert F(\theta)\rvert$
0°	0.5	0°	0	0
5	0.5	0	0.0171	0.1218
10	0.5	0	0.0677	0.2425
15	0.5	0	0.1492	0.3599
20	0.5	0	0.2549	0.4705
25	0.5	0	0.3718	0.5682
30	0.5	0	0.4790	0.6449
35	0.5	0	0.5497	0.6909
40^-	0.5	0	0.5581	0.6961
40^+	0.3	-10°	0.4321	0.6125
45	0.3	-10	0.4161	0.6011
50	0.3	-10	0.3755	0.5710
55	0.3	-10	0.3453	0.5476
60	0.3	-10	0.3743	0.5701
65	0.3	-10	0.5062	0.6630
70^-	0.3	-10	0.7544	0.8094
70^+	0.1	-90°	1.0398	0.9502
75	0.1	-90	1.1286	0.9899
80^-	0.1	-90	1.1516	1.0
80^+	0.5	-180°	0.7647	0.8149
85	0.5	-180	0.3932	0.5843
90	0.5	-180	0.2500	0.4659

$|F(\theta)|$ is found by dividing $|F_u(\theta)|^2$ by 1.1516 and taking the square root.

5.5-6 (con't)

The pattern plot is



(b) For comparison the pattern of a short dipole above a perfect ground plane is show in the pattern above as a dashed curve. This pattern follows from Prob. 5.5-4a or Fig. 5-37b ($n=\infty$).

5.5-7

From (2-19) $Z_{in,mono} = \frac{1}{2} Z_{in,dipole} = \frac{1}{2}(70+j0) = 35$ ohms since a good ground plane is present.

The radiation efficiency is

$$e = \frac{R_r}{R_r + R_{ohmic}} \Rightarrow R_{ohmic} = R_r\left(\frac{1}{e}-1\right) = 35\left(\frac{1}{0.97}-1\right) = 1.08\,\Omega$$

From Fig. 5-39 at 1MHz for $R_{ohmic} \approx 1$ ohm $\ell/\lambda = 0.2$

So $\ell = 0.2\lambda = 0.2(300m) = \boxed{60\ m}$ = radial lengths

5.6-1 $\quad L = 6\lambda$

From (5-58)

$$\frac{1}{K}F(\theta) = \sin\theta \frac{\sin\left[\frac{\beta L}{2}(1-\cos\theta)\right]}{\frac{\beta L}{2}(1-\cos\theta)} = \sin\theta \frac{\sin[6\pi(1-\cos\theta)]}{6\pi(1-\cos\theta)}$$

for $L = 6\lambda$

At $\theta = 20.1°$

$$\frac{1}{K}F(\theta_m = 20.1°) = 0.27299$$

But at $\theta = 20.2°$ $\quad \frac{1}{K}F(\theta = 20.2°) = 0.27298$

And at $\theta = 20.3°$ $\quad \frac{1}{K}F(\theta = 20.3°) = 0.27294$

Thus pattern is maximum at $\theta = 20.1°$

5.6-2 Traveling-wave long wire

L/λ	$\theta_m = \cos^{-1}\left(1-\frac{0.371\lambda}{L}\right)$ from (5-59)	θ_m from Fig. 5-42
1	51°	47°
3	28.8°	28°
5	22.2°	22°
6	20.25°	20°
10	15.65°	16°

5.6-3

From (4-8) and (5-58)

$$E_\theta = j\omega\mu \frac{e^{-j\beta r}}{4\pi r} I_m L \sin\theta \frac{\sin\left[\frac{\beta L}{2}(1-\cos\theta)\right]}{\frac{\beta L}{2}(1-\cos\theta)}$$

Using (1-127)

$$P_r = \frac{1}{2\eta}\iint |E_\theta|^2 r^2 \sin\theta\, d\theta\, d\phi$$

5.6-3 (con't)

$$P_r = \frac{1}{2\eta}\left(\frac{\omega\mu}{4\pi} I_m L\right)^2 \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi} \sin^3\theta \left[\frac{\sin\left(\frac{\beta L}{2}(1-\cos\theta)\right)}{\frac{\beta L}{2}(1-\cos\theta)}\right]^2 d\theta}_{\mathcal{I}}$$

$$= \frac{\omega^2\mu^2\, 2\pi}{2\eta\, 16\pi^2} I_m^2 L^2 \mathcal{I} = \frac{1}{16\pi} \frac{\omega^2\mu^2}{\eta} I_m^2 L^2 \mathcal{I}$$

Let $\tau = \frac{\beta L}{2}(1-\cos\theta)$ then $d\tau = \frac{\beta L}{2}\sin\theta\, d\theta$ and

$\cos\theta = 1 - \frac{2\tau}{\beta L}$ $\quad \sin^2\theta = 1-\cos^2\theta = 1-(1-\frac{2\tau}{2L})^2 = \frac{4}{\beta L}\tau - \frac{4}{(\beta L)^2}\tau^2$

Then

$$\mathcal{I} = \int_0^{\beta L} \left[\frac{4}{\beta L}\tau - \frac{4}{(\beta L)^2}\tau\right] \frac{\sin^2\tau}{\tau^2} \frac{d\tau}{\frac{\beta L}{2}}$$

$$= \frac{8}{(\beta L)^2}\left\{\int_0^{\beta L} \frac{\sin^2\tau}{\tau} d\tau - \int_0^{\beta L} \sin^2\tau\, d\tau \frac{1}{\beta L}\right\}$$

$$= \frac{4}{(\beta L)^2}\left\{\int_0^{2\beta L} \frac{1-\cos t}{t} dt - \int_0^{\beta L} (1-\cos 2\tau) d\tau \frac{1}{\beta L}\right\} \text{ where } t = 2\tau$$

$$= \frac{4}{(\beta L)^2}\left\{\mathrm{Cin}(2\beta L) - \frac{1}{\beta L}\left(\beta L - \frac{\sin 2\beta L}{2}\right)\right\} \quad \text{using (F-15)}$$

$$= \frac{4}{(\beta L)^2}\left[0.5772 + \ln(2\beta L) - \mathrm{Ci}(2\beta L) - 1 + \frac{\sin 2\beta L}{2\beta L}\right] \quad \text{using (F-16)}$$

And $\ln 2\beta L = \ln 4\pi\frac{L}{\lambda} = \ln 4\pi + \ln\frac{L}{\lambda} = 2.5310 + \ln\frac{L}{\lambda}$

So

$$P_r = \frac{1}{16\pi} I_m^2 L^2 \frac{\omega^2\mu^2}{\eta} \frac{4}{\omega^2\mu\epsilon L^2}\left[(0.5772 - 1 + 2.5310) + \ln\frac{L}{\lambda} - \mathrm{Ci}(2\beta L) + \frac{\sin 2\beta L}{2\beta L}\right]$$

$$= I_m^2 \frac{\mu/\epsilon}{\sqrt{\mu/\epsilon}} \frac{1}{4\pi}[\ ] = I_m^2 \sqrt{\mu/\epsilon} \frac{1}{4\pi}[\ ] = \frac{120\pi}{4\pi} I_m^2 [\ ]$$

$$= 30 I_m^2 \left[2.108 + \ln\frac{L}{\lambda} - \mathrm{Ci}(2\beta L) + \frac{\sin 2\beta L}{2\beta L}\right] \quad \text{as required.}$$

5.6-4

(a) $$D = \frac{U_m}{P_r/4\pi} = \frac{4\pi}{P_r}\frac{1}{2\eta}|E_\theta|^2 r^2$$

$$= \frac{4\pi}{P_r}\frac{1}{2\eta}\frac{\omega^2\mu^2}{16\pi^2} I_m^2 L^2 \sin^2\theta_m \frac{\sin^2\left[\frac{\beta L}{2}(1-\cos\theta_m)\right]}{\left[\frac{\beta L}{2}(1-\cos\theta_m)\right]^2} \quad \text{using } E_\theta \text{ from Prob. 5.6-3}$$

$$= \frac{4\pi}{P_r}\frac{1}{2\eta}\frac{\omega^2\mu^2\, 4}{16\pi^2\, \omega^2\mu\,\epsilon} I_m^2 \left[\frac{\sin\theta_m}{1-\cos\theta_m}\right]^2 \sin^2\left[\frac{\beta L}{2}(1-\cos\theta_m)\right]$$

now

$$\frac{\sin^2\theta_m}{(1-\cos\theta_m)^2} = \frac{\left(2\sin\frac{\theta_m}{2}\cos\frac{\theta_m}{2}\right)^2}{2\sin^2\frac{\theta_m}{2}} = \cot^2\frac{\theta_m}{2}$$

5.6-4 (con't)

From (5-59) $1-\cos\theta_m = \dfrac{0.371}{L/\lambda} \Rightarrow \dfrac{\beta L}{2}(1-\cos\theta_m) = \pi 0.371$

and $\sin\left[\dfrac{\beta L}{2}(1-\cos\theta_m)\right] = \sin(0.371\pi) = 0.919$

Using P_r from Prob. 5.6-3

$$D = \frac{\frac{1}{2\pi} I_m^2 \sqrt{\mu/\epsilon}\, \cot^2\left[\frac{1}{2}\cos^{-1}\left(1-\frac{0.371}{L/\lambda}\right)\right](0.919)^2}{30 I_m^2\left[2.108 + \ln\frac{L}{\lambda} - C_i(2\beta L) + \frac{\sin 2\beta L}{2\beta L}\right]}$$

and we use $\sqrt{\frac{\mu}{\epsilon}} = 120\pi$

giving

$$D = \frac{1.69 \cot^2\left[\frac{1}{2}\cos^{-1}\left(1-\frac{0.371}{L/\lambda}\right)\right]}{2.108 + \ln\frac{L}{\lambda} - C_i(2\beta L) + \frac{\sin 2\beta L}{2\beta L}}$$

as required.

(b)

L/λ	D	D(dB)
2	5.90	7.71
5	11.79	10.71
10	20.27	13.07
20	35.40	15.49

5.6-5

(a) Using P_r from Prob. 5.6-3

$$R_r = \frac{2P_r}{I_m^2} = 60\left[2.108 + \ln\frac{L}{\lambda} - C_i(2\beta L) + \frac{\sin 2\beta L}{2\beta L}\right]$$

(b)

L/λ	R_r
2	168 ohms
5	223
10	264.6
20	306.2

5.6-6

The program WAVEANT was used to produce this pattern.

WAVEANT

TRAVELING WAVE LINEAR ANTENNA PROGRAM

ANTENNA LENGTH= 8.0000 WAVELENGTHS

VELOCITY FACTOR= 1.0000

ALL PATTERN VALUES HAVE BEEN DIVIDED BY EMAX= 0.237049

5.6-6 (con't)

```
C * * * * * * * * * * * * * WAVEANT * * * * * * * * * * * * * *
C THIS PROGRAM CONPUTES THE PATTERN OF A TRAVELING WAVE LINEAR
C    ANTENNA ALONG THE Z-AXIS
C
C L=LENGTH OF THE ANTENNA/FREE SPACE WAVELENGTH
C P=VELOCITY FACTOR
C THETA=ANGLE FROM ANTENNA
C
      REAL L,P, DATA(360)
C
      PI=3.14159265
      DTR=PI/180.
 1    CONTINUE
      READ(5,80,END=100) L,P
 80   FORMAT(2F10.5)
      WRITE(6,90)
 90   FORMAT(1H1,20X,'WAVEANT'//' TRAVELING WAVE LINEAR ANTENNA PROGRAM'
     1)
      WRITE(6,91) L,P
 91   FORMAT(1H ,/' ANTENNA LENGTH=',F8.4,' WAVELENGTHS'//
     1 ' VELOCITY FACTOR=',F8.4)
C
      EMAX=0.
      DO 10 I=1,360
      THETA=I*DTR
      ARG=PI*L*(1./P-COS(THETA))
      IF(ARG.EQ.0.) DATA(I)=SIN(THETA)
      IF(ARG.NE.0.) DATA(I)=SIN(THETA)*SIN(ARG)/ARG
      E=ABS(DATA(I))
      IF(E.GT.EMAX) EMAX=E
 10   CONTINUE
      DO 20 I=1,360
 20   DATA(I)=DATA(I)/EMAX
      WRITE(6,92) EMAX
 92   FORMAT(//' ALL PATTERN VALUES HAVE BEEN DIVIDED BY EMAX=',F10.6)
C
      CALL PPLOT(DATA)
      GO TO 1
 100  STOP
      END
```

5.6-7

The current $I_t(z) = I_m e^{-(a+j\beta_0)z}$ in (4-1) gives

$$f_{un} = I_m \int_{-\frac{L}{2}}^{L/2} e^{-(a+j(\beta_0-\beta\cos\theta))z'}\,dz'$$

$$= I_m \frac{e^{(-a+(\beta\cos\theta-\beta_0))\frac{L}{2}} - e^{-(-a+(\beta\cos\theta-\beta_0))\frac{L}{2}}}{-a+j(\beta\cos\theta-\beta_0)}$$

$$= I_m \frac{L}{2} 2 \frac{\sinh\left[(-a+j(\beta\cos\theta-\beta_0))\frac{L}{2}\right]}{(-a+j(\beta\cos\theta-\beta_0))\frac{L}{2}}$$

$$= I_m L \frac{\sinh\left[-\frac{aL}{2}-j\frac{\beta L}{2}\left(\frac{\beta_0}{\beta}-\cos\theta\right)\right]}{-\frac{aL}{2}-j\frac{\beta L}{2}\left(\frac{\beta_0}{\beta}-\cos\theta\right)} = I_m L \frac{\sinh\left[\frac{aL}{2}+j\frac{\beta L}{2}\left(\frac{1}{p}-\cos\theta\right)\right]}{\frac{aL}{2}+j\frac{\beta L}{2}\left(\frac{1}{p}-\cos\theta\right)}$$

Then from (4-1), the total pattern is

$$F(\theta) = K\sin\theta \frac{\sinh\left[\frac{aL}{2}+j\frac{\beta L}{2}\left(\frac{1}{p}-\cos\theta\right)\right]}{\frac{aL}{2}+j\frac{\beta L}{2}\left(\frac{1}{p}-\cos\theta\right)}$$

5.6-7 (con't)

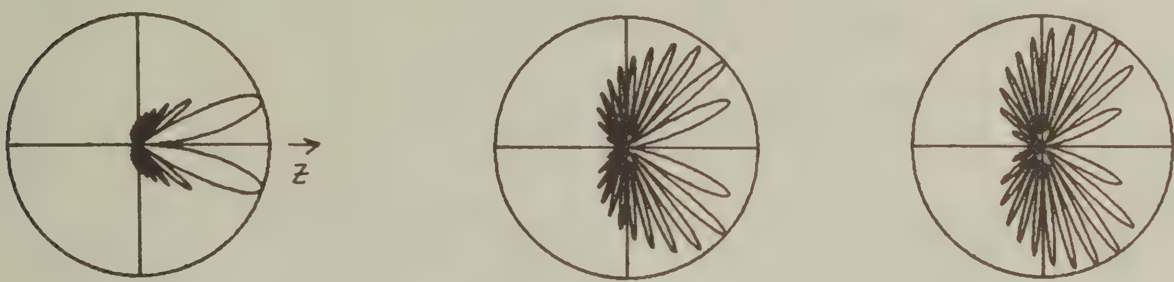
(b) For $a=0$ and $p=1$ this reduces to

$$F(\theta)=K\sin\theta\,\frac{\sinh\left[j\frac{\beta L}{2}\left(\frac{1}{1}-\cos\theta\right)\right]}{\frac{\beta L}{2}(1-\cos\theta)}=K\sin\theta\,\frac{\sin\left[\frac{\beta L}{2}(1-\cos\theta)\right]}{\frac{\beta L}{2}(1-\cos\theta)}$$

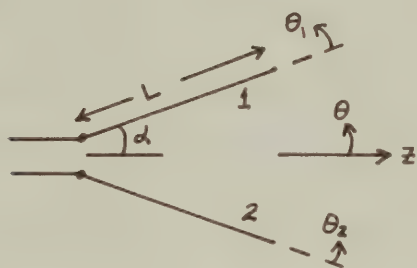
which is (5-58).

(c) Using WAVEANT

$L=6\lambda$ $p=1.0$ $\qquad$ $L=6\lambda$ $p=0.75$ $\qquad$ $L=6\lambda$ $p=0.5$



5.6-8



The radiation integral for side 1 with zero current phase at the vertex is

$$f_{un}=\int_0^L I_m e^{-j\beta z_1}e^{j\beta z_1\cos\theta_1}dz_1 = I_m\,\frac{e^{j\beta(-1+\cos\theta_1)L}-1}{j\beta(-1+\cos\theta_1)}$$

$$=I_m e^{j\frac{\beta L}{2}(-1+\cos\theta_1)}\,\frac{e^{j\frac{\beta L}{2}(-1+\cos\theta)}-e^{-j\frac{\beta L}{2}(-1+\cos\theta_1)}}{j\beta(-1+\cos\theta_1)}$$

$$=L I_m e^{j\frac{\beta L}{2}(-1+\cos\theta_1)}\,\frac{\sin\left[\frac{\beta L}{2}(-1+\cos\theta_1)\right]}{\frac{\beta L}{2}(-1+\cos\theta_1)}$$

But $\theta_1=\theta-\alpha$ and $\theta_2=\theta+\alpha$

So $$f_{un_1}=L I_m e^{j\frac{\beta L}{2}(-1+\cos(\theta-\alpha))}\,\frac{\sin\left[\frac{\beta L}{2}(-1+\cos(\theta-\alpha))\right]}{\frac{\beta L}{2}(-1+\cos(\theta-\alpha))}$$

and similarly for leg #2

$$f_{un_2}=L I_m e^{j\frac{\beta L}{2}(-1+\cos(\theta+\alpha))}\,\frac{\sin\left[\frac{\beta L}{2}(-1+\cos(\theta+\alpha))\right]}{\frac{\beta L}{2}(-1+\cos(\theta+\alpha))}$$

5.6-8 (con't)

Then the complete radiation pattern is

$$F_v(\theta) = K_v[F_1(\theta) - F_2(\theta)]$$

where

$$F_1(\theta) = e^{j\frac{\beta L}{2}(-1+\cos(\theta-\alpha))}\sin(\theta-\alpha)\,\frac{\sin[\frac{\beta L}{2}(-1+\cos(\theta-\alpha))]}{\frac{\beta L}{2}(-1+\cos(\theta-\alpha))}$$

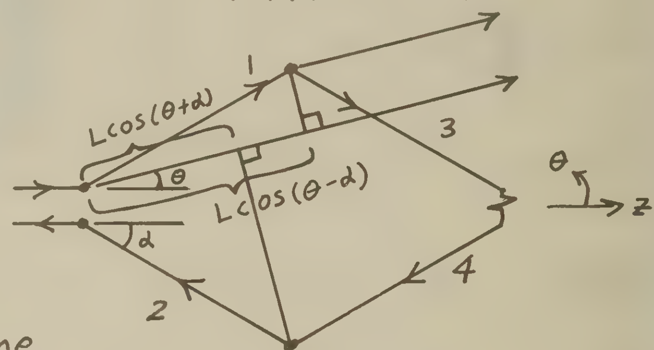
$$F_2(\theta) = e^{j\frac{\beta L}{2}(-1+\cos(\theta+\alpha))}\sin(\theta+\alpha)\,\frac{\sin[\frac{\beta L}{2}(-1+\cos(\theta+\alpha))]}{\frac{\beta L}{2}(-1+\cos(\theta+\alpha))}$$

The minus sign on F_2 arises because of the oppositely directed current on leg #2.

(b) See Fig. 5-43 and program VEE in Prob. 5.6-9.

5.6-9

(a) The phase reference points for each side of the rhombic are indicated by bold dots.



The total pattern of the rhombic consists of the four line source patterns of each leg weighted in phase by the phase shift of the end of each relative to the vertex:

$$F_R(\theta) = K_R[F_1(\theta) - F_2(\theta) + e^{-j\beta L}F_3(\theta) - e^{-j\beta L}F_4(\theta)]$$

where

$$F_3(\theta) = e^{j\beta L\cos(\theta-\alpha)}F_2(\theta) \qquad F_4(\theta) = e^{j\beta L\cos(\theta+\alpha)}F_1(\theta)$$

(b) See Fig. 5-44 for the plots

The program VEE was used to generate vee & rhombic patterns. It follows

```
C * * * * * * * * * * * * * VEE * * * * * * * * * * * * * * *
C THIS PROGRAM COMPUTES THE PATTERN OF VEE AND RHOMBIC ANTENNAS
C   THE PATTERN IS IN THE PLANE OF THE ANTENNA
C
C L=LENGTH OF ONE LEG/WAVELENGTH
C
      REAL L,DATAV(360),DATAR(360)
      COMPLEX F,E3,E4,IMAG
C
      PI=3.14159265
      DTR=PI/180.
      IMAG=(0.,1.)
C
 1    CONTINUE
      READ(5,80,END=100) L,ALPHA
 80   FORMAT(2F10.6)
```

5.6-9 (con't)

```
      WRITE(6,90)
   90 FORMAT(1H1,20X,'VEE'//' VEE AND RHOMBIC ANTENNA PROGRAM')
      WRITE(6,91) L
   91 FORMAT(1H ,' THE LEG LENGTH=L=',F10.6)
C
      ALPHAD=ALPHA
      ALPHA=ALPHAD*DTR
      WRITE(6,93) ALPHAD
   93 FORMAT(1H ,' ALPHA=',F10.6,' DEGREES')
      EMAX=0.
      FMAX=0.
      DO 10 I=1,360
      THETA=I*DTR
      ARG1=PI*L*(1.-COS(THETA-ALPHA))
      ARG2=PI*L*(1.-COS(THETA+ALPHA))
      SINC1=1.0
      SINC2=1.0
      IF(ARG1.NE.0.) SINC1=SIN(ARG1)/ARG1
      IF(ARG2.NE.0.) SINC2=SIN(ARG2)/ARG2
      E1=SIN(THETA-ALPHA)*SINC1
      E2=-SIN(THETA+ALPHA)*SINC2
      E1=E1*CEXP(-IMAG*ARG1)
      E2=E2*CEXP(-IMAG*ARG2)
      E=E1+E2
      E3=-CEXP(IMAG*2.*PI*L*COS(THETA-ALPHA))*E2
      E4=-CEXP(IMAG*2.*PI*L*COS(THETA+ALPHA))*E1
      F=E1+E2+CEXP(-IMAG*2.*PI*L)*(E3+E4)
      DATAV(I)=ABS(E)
      DATAR(I)=CABS(F)
      IF(DATAV(I).GT.EMAX) EMAX=DATAV(I)
      IF(DATAR(I).GT.FMAX) FMAX=DATAR(I)
   10 CONTINUE
      DO 20 I=1,360
      DATAV(I)=DATAV(I)/EMAX
   20 DATAR(I)=DATAR(I)/FMAX
C
      WRITE(6,94)
   94 FORMAT(1H1,' THE PATTERN OF THE VEE')
      WRITE(6,95) EMAX

   95 FORMAT(1H ,' ALL PATTERN VALUES HAVE BEEN DIVIDED BY',F10.6)
      CALL PPLOT(DATAV)
      WRITE(6,96)
   96 FORMAT(1H1,' THE PATTERN OF THE RHOMBIC')
      WRITE(6,95) FMAX
      CALL PPLOT(DATAR)
      GO TO 1
  100 STOP
      END
```

5.6-10

From (5-61) $h = \dfrac{\lambda}{4\sin\alpha} = \dfrac{\lambda}{4\sin 20^\circ} = \boxed{0.731\lambda}$

From (5-62) $L = \dfrac{0.371\lambda}{\sin^2\alpha} = \dfrac{0.371\lambda}{\sin^2 20^\circ} = \boxed{3.17\lambda}$

5.7-1

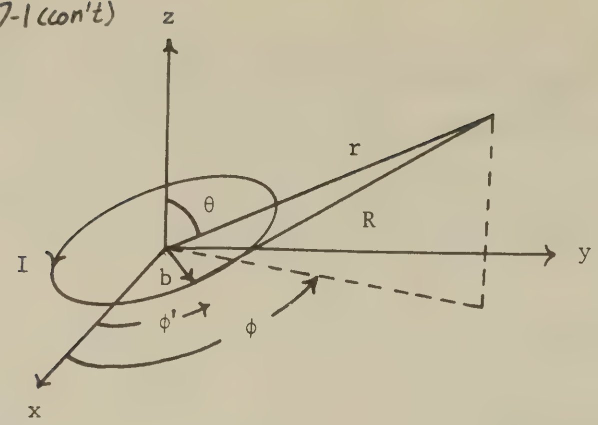
From (1-99)

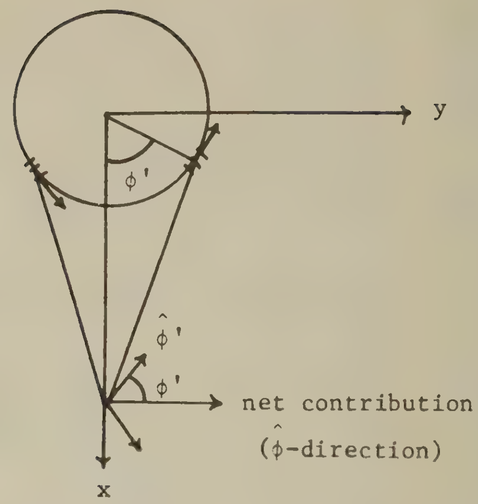
$$\vec{A} = \frac{e^{-j\beta r}}{4\pi r}\int_0^{2\pi} I_o\,\hat{\phi}'\,e^{j\beta\hat{r}\cdot\vec{r}\,'}\,b\,d\phi'$$

And

$$\hat{r}\cdot\vec{r}\,' = (\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\phi\,\hat{z})\cdot(b\cos\phi'\hat{x} + b\sin\phi'\hat{y})$$
$$= b\sin\theta(\cos\phi\cos\phi' + \sin\phi\sin\phi') = b\sin\theta\cos(\phi-\phi')$$

5.7-1 (con't)





Since the loop current is symmetric with ϕ (does not vary with ϕ), the radiated fields will be independent of ϕ. For simplicity choose $\underline{\phi = 0}$. Then

$$\hat{r} \cdot \vec{r}\,' = b \sin\theta \cos\phi'$$

From the figure we see that there will only be a ϕ-component, since current elements equidistant to the field point will pairwise yield only a net ϕ-component. So

$$\vec{A} = A_\phi \hat{\phi} = \hat{\phi} \frac{e^{-j\beta r}}{4\pi r} I_o b \int_0^{2\pi} \hat{\phi} \cdot \hat{\phi}' e^{j\beta \hat{r} \cdot \vec{r}\,'} d\phi'$$

And $\hat{\phi} \cdot \hat{\phi}' = \cos\phi'$ (see figure); so

$$\vec{A} = \hat{\phi} \frac{e^{-j\beta r}}{4\pi r} I_o b \int_0^{2\pi} \cos\phi' e^{j\beta b \sin\theta \cos\phi'} d\phi' \qquad \text{QED.}$$

(b)

From (F-7) with $n=1$,

$$J_1(x) = \frac{j^{-1}}{2\pi} \int_0^{2\pi} e^{jx\cos\alpha} \cos\alpha \, d\alpha$$

So

$$\int_0^{2\pi} e^{j\beta b \sin\theta \cos\phi'} \cos\phi' \, d\phi' = 2\pi j \, J_1(\beta b \sin\theta)$$

Thus

$$\vec{A} = \hat{\phi} \frac{e^{-j\beta r}}{4\pi r} I_o b \, 2\pi j \, J_1(\beta b \sin\theta) = \hat{\phi} j \frac{e^{-j\beta r}}{2r} I_o b J_1(\beta b \sin\theta)$$

Now

$$\vec{E} = -j\omega\mu \vec{A} \text{ (transverse only)} = -j\omega\mu \hat{\phi} j \frac{e^{-j\beta r}}{2r} I_o b J_1(\beta b \sin\theta)$$

$$\boxed{\vec{E} = \frac{\omega\mu I_o b e^{-j\beta r}}{2r} J_1(\beta b \sin\theta)}$$

5.7-1 (con't)

(c) For a small loop $b \ll \lambda$, or $\frac{b}{\lambda} \ll 1$, or $\beta b \ll 1$, or $\beta b \sin\theta \ll 1$.

Then
$$J_1(\beta b \sin\theta) \approx \frac{\beta b \sin\theta}{2}$$

So
$$\vec{E} \approx \hat{\phi}\, \frac{\omega\mu I_0 b e^{-j\beta r}}{2r}\, \frac{\beta b \sin\theta}{2} = \hat{\phi}\, \omega\mu\beta\pi b^2 \frac{I_0 e^{-j\beta r}}{4\pi r} \sin\theta$$
and $\omega\mu\beta = \omega\sqrt{\mu\epsilon}\,\sqrt{\frac{\mu}{\epsilon}}\,\beta = \eta\beta^2$

Thus
$$\vec{E} = \hat{\phi}\,\eta\beta^2 S\, \frac{I_0 e^{-j\beta r}}{4\pi r} \sin\theta$$
which is the small loop result of (2-53).

5.7-2

From (5-66)
$$\vec{A} = \frac{e^{-j\beta r}}{4\pi r} I_0 \left[-\hat{x}\, 2\cos\left(\frac{\pi}{4}\sin\theta\sin\phi\right) \mathcal{J}_1 + \hat{y}\, 2j \sin\left(\frac{\pi}{4}\sin\theta\cos\phi\right) \mathcal{J}_2 \right]$$
where
$$\mathcal{J}_1 = \int_{-\lambda/8}^{\lambda/8} \cos(\beta x') e^{j\beta x' \sin\theta\cos\phi}\, dx', \quad \mathcal{J}_2 = \int_{-\lambda/8}^{\lambda/8} \sin(\beta y') e^{j\beta x' \sin\theta\sin\phi}\, dy'$$

Consider $\mathcal{J}_1$ first. Expand $\cos\beta x'$ using (C-7)
$$\mathcal{J}_1 = \frac{1}{2}\left\{ \int_{-\lambda/8}^{\lambda/8} e^{j\beta(1+\sin\theta\cos\phi)x'}\, dx' + \int_{-\lambda/8}^{\lambda/8} e^{j\beta(-1+\sin\theta\cos\phi)x'}\, dx' \right\}$$
$$= \frac{1}{2j\beta}\left[\frac{e^{j\frac{\pi}{4}(1+\sin\theta\cos\phi)} - e^{-j\frac{\pi}{4}(1+\sin\theta\cos\phi)}}{1+\sin\theta\cos\phi} + \frac{e^{j\frac{\pi}{4}(-1+\sin\theta\cos\phi)} - e^{-j\frac{\pi}{4}(-1+\sin\theta\cos\phi)}}{-1+\sin\theta\cos\phi} \right]$$
$$= \frac{1}{\beta}\left[\frac{\sin\left[\frac{\pi}{4}(1+\cos\gamma)\right]}{1+\cos\gamma} + \frac{\sin\left[\frac{\pi}{4}(-1+\cos\gamma)\right]}{-1+\cos\gamma} \right] \quad \text{where } \cos\gamma = \sin\theta\cos\phi$$
$$= \frac{1}{\beta(-1+\cos^2\gamma)}\left\{ (-1+\cos\gamma)\sin\left[\frac{\pi}{4}(1+\cos\gamma)\right] + (1+\cos\gamma)\sin\left[\frac{\pi}{4}(-1+\cos\gamma)\right] \right\}$$
$$= \frac{-1}{\beta\sin^2\gamma}\left\{ \cos\gamma\left[\sin\left(\frac{\pi}{4}(1+\cos\gamma)\right) + \sin\left(\frac{\pi}{4}(-1+\cos\gamma)\right)\right] - \left[\sin\left(\frac{\pi}{4}(1+\cos\gamma)\right) - \sin\left(\frac{\pi}{4}(-1+\cos\gamma)\right)\right] \right\}$$
$$= \frac{-1}{\beta\sin^2\gamma}\left[\cos\gamma \cdot 2\, \sin\left(\frac{\pi}{4}\cos\gamma\right)\cos\frac{\pi}{4} - 2\cos\left(\frac{\pi}{4}\cos\gamma\right)\sin\frac{\pi}{4} \right] \quad \text{using (B-6) \& (B-7)}$$
$$= \frac{-\sqrt{2}}{\beta\sin^2\gamma}\left[\cos\gamma\, \sin\left(\frac{\pi}{4}\cos\gamma\right) - \cos\left(\frac{\pi}{4}\cos\gamma\right) \right]$$

Now
$$\mathcal{J}_2 = \frac{1}{2j}\left\{ \int_{-\lambda/8}^{\lambda/8} e^{j\beta(1+\sin\theta\sin\phi)y'}\, dy' - \int_{-\lambda/8}^{\lambda/8} e^{j\beta(-1+\sin\theta\sin\phi)y'}\, dy' \right\} \quad \text{using (C-6)}$$
$$= \frac{-1}{2\beta}\left[\frac{e^{j\frac{\pi}{4}(1+\cos\Omega)} - e^{-j\frac{\pi}{4}(1+\cos\Omega)}}{1+\cos\Omega} - \frac{e^{j\frac{\pi}{4}(-1+\cos\Omega)} - e^{-j\frac{\pi}{4}(-1+\cos\Omega)}}{-1+\cos\Omega} \right]$$
where $\cos\Omega = \sin\theta\sin\phi$

5.7-2 (con't)

$$\mathcal{J}_z = \frac{-j}{\beta}\left\{\frac{\sin[\frac{\pi}{4}(1+\cos\Omega)]}{1+\cos\Omega} - \frac{\sin[\frac{\pi}{4}(-1+\cos\Omega)]}{-1+\cos\Omega}\right\}$$

$$= \frac{-j}{\beta(-1+\cos^2\Omega)}\left\{(-1+\cos\Omega)\sin[\tfrac{\pi}{4}(1+\cos\Omega) - (1+\cos\Omega)\sin[\tfrac{\pi}{2}(-1+\cos\Omega)]\right\}$$

$$= \frac{j}{\beta\sin^2\Omega}\left\{\cos\Omega[\sin(\tfrac{\pi}{4}(1+\cos\Omega)) - \sin(\tfrac{\pi}{4}(-1+\cos\Omega))]\right.$$
$$\left. - [\sin(\tfrac{\pi}{4}(1+\cos\Omega)) + \sin(\tfrac{\pi}{4}(-1+\cos\Omega))]\right\}$$

$$= \frac{j}{\beta\sin^2\Omega}\left[\cos\Omega\ 2\cos(\tfrac{\pi}{4}\cos\Omega)\sin\tfrac{\pi}{4} - 2\sin(\tfrac{\pi}{4}\cos\Omega)\cos\tfrac{\pi}{4}\right] \quad \text{using (B-5) + (B-6)}$$

$$= \frac{j\sqrt{2}}{\beta\sin^2\Omega}\left[\cos\Omega\cos(\tfrac{\pi}{4}\cos\Omega) - \sin(\tfrac{\pi}{4}\cos\Omega)\right]$$

Combining the results:

$$\vec{A} = \frac{e^{-j\beta r}}{4\pi r} I_o\left\{-\hat{x}\,2\cos(\tfrac{\pi}{4}\cos\Omega)\frac{-\sqrt{2}}{\beta\sin^2\gamma}\left[\cos\gamma\sin(\tfrac{\pi}{4}\cos\gamma) - \cos(\tfrac{\pi}{4}\cos\gamma)\right]\right.$$
$$\left. + \hat{y}\,j2\sin(\tfrac{\pi}{4}\cos\gamma)\frac{j\sqrt{2}}{\beta\sin^2\Omega}\left[\cos\Omega\cos(\tfrac{\pi}{4}\cos\Omega) - \sin(\tfrac{\pi}{4}\cos\Omega)\right]\right\}$$

$$= \frac{e^{-j\beta r}}{4\pi r}\frac{2\sqrt{2}I_o}{\beta}\left\{\hat{x}\frac{\cos(\frac{\pi}{4}\cos\Omega)}{\sin^2\gamma}\left[\cos\gamma\sin(\tfrac{\pi}{4}\cos\gamma) - \cos(\tfrac{\pi}{4}\cos\gamma)\right]\right.$$
$$\left. - \hat{y}\frac{\sin(\frac{\pi}{4}\cos\gamma)}{\sin^2\Omega}\left[\cos\Omega\cos(\tfrac{\pi}{4}\cos\Omega) - \sin(\tfrac{\pi}{4}\cos\Omega)\right]\right\}$$

which is (5-67).

5.7-3

Eqn. (2-64) is

$$L = \frac{\mu}{\pi}\left[l_2\cosh^{-1}\frac{l_1}{d} + l_1\cosh^{-1}\frac{l_2}{d}\right]$$

For a square loop $l_1 = l_2 = \frac{p}{4}$ where p = perimeter and $d = 2a$

So $L = \frac{\mu}{\pi}\frac{p}{4}\,2\cosh^{-1}\frac{p/4}{2a}$

Thus
$$X_{in} = \omega L = 2\pi\frac{c}{\lambda}\mu\frac{p}{2\pi}\cosh^{-1}\frac{p}{8a}$$
$$= \eta\frac{p}{\lambda}\cosh^{-1}\frac{p}{8a} \qquad \text{since } c\mu = \frac{\mu}{\sqrt{\mu\epsilon}} = \sqrt{\frac{\mu}{\epsilon}} = \eta$$

For $p = 0.2\lambda$ and $a = 0.001\lambda$

$$X_{in} = 120\pi(0.2)\cosh^{-1}\frac{0.2}{0.008} = \boxed{294.9 \text{ ohms}}$$

Comparing to Fig. 5-47b for the exact result

$$X_{in} = 227 \text{ ohms}$$

CHAPTER 6

6.1-1

$\lambda = 17.6$ cm @ 1.7 GHz

$A = \text{length} = 78.7\,\text{cm} \quad A/\lambda = 4.47 \qquad \alpha = 11.7°$

$D = \text{diameter} = 4.84\,\text{cm} \quad D/\lambda = 0.275$

$C = \pi D = 15.2\,\text{cm} \quad C/\lambda = 0.864$

$S = \text{spacing} = C \tan\alpha = 15.2 \tan 11.7° = 3.15\,\text{cm} \quad S/\lambda = 0.179$

$$N = A/S = \frac{78.7}{3.15} = \boxed{25 \text{ turns}}$$

From (6-26)

$$D = \text{directivity} = 15\left(\frac{C}{\lambda}\right)^2 N\left(\frac{S}{\lambda}\right) = 15(0.864)^2 25(0.179)$$
$$= 50.11 = \boxed{17.0\,\text{dB}}$$

From (6-24)

$$HP = \frac{52°}{\left(\frac{C}{\lambda}\right)\sqrt{N S/\lambda}} = \frac{52°}{0.864\sqrt{25(0.179)}} = \boxed{28.5°}$$

From (6-27)

$$|AR| = \frac{2N+1}{2N} = \frac{51}{50} = \boxed{1.02}$$

Ref.: Communications Designers Digest, January 1970.

6.1-2

$HP = 39° \qquad \alpha = 12.5°$

(a) $\lambda = 0.632$ m @ 475 MHz

$C/\lambda = 1$ at this center frequency where $HP = 39°$

Solving (6-24) for A/λ

$$A/\lambda = \left[\frac{52°}{C/\lambda \; HP}\right]^2 = \left[\frac{52°}{(1)(39°)}\right]^2 = 1.78$$

Then $S/\lambda = C/\lambda \tan\alpha = (1)\tan(12.5°) = 0.222$

And $A = NS$, $A/\lambda = N S/\lambda$, $N = \dfrac{A/\lambda}{S/\lambda} = \dfrac{1.78}{0.222} = \boxed{8 \text{ turns}}$

(b) From (6-26)

$$D = 15\left(\frac{C}{\lambda}\right)^2 N \frac{S}{\lambda} = 15(1)^2\, 8\,(0.222) = 26.64 = \boxed{14.25\,\text{dB}}$$

6.1-2 (con't)

(c) From (6-27)

$$|AR| = \frac{2N+1}{2N} = \frac{17}{16} = \boxed{1.06}$$

(d) At the low frequency end of the band, from (6-12),

$$C = \tfrac{3}{4}\lambda_L \Rightarrow \lambda_L = \tfrac{4}{3}C = \tfrac{4}{3}\lambda = \tfrac{4}{3}(0.632) = 0.843\,m$$

$$f_L = \frac{c}{\lambda_L} = \frac{3\times10^8}{0.843} = \boxed{356\ MHz}$$

Also $C = \tfrac{4}{3}\lambda_u \Rightarrow \lambda_u = \tfrac{3}{4}C = \tfrac{3}{4}(0.632) = 0.474\,m$

$$f_u = \frac{c}{\lambda_u} = \frac{3\times10^8}{0.474} = \boxed{633\ MHz}$$

(e) From (6-28) $R_{in} = 140\frac{C}{\lambda}$ ohms

At the design frequency $C/\lambda = 1$, so $\boxed{R_{in} = 140\ ohms}$

At the low end of the band $R_{in} = 140(\tfrac{3}{4}) = \boxed{105\ ohms}$

At the upper end of the band $R_{in} = 140(\tfrac{4}{3}) = \boxed{186.7\ ohms}$

6.1-3

$N=6$ $A = 118\ cm$ $D = 22.3\ cm$ (Taco Model G-1215)

(a)

$$C = \pi D = \pi(22.3) = 70.1\ cm \qquad S = \frac{A}{N} = \frac{118}{6} = 19.7\,cm$$

$$\alpha = \tan^{-1}\frac{S}{C} = \tan^{-1}\frac{19.7}{70.1} = \boxed{15.7^\circ}$$

(b)

$$G \approx D = 15\left(\frac{C}{\lambda}\right)^2 N \frac{S}{\lambda}$$

@ 300 MHz $\lambda = 1\,m$ $G = 15\left(\frac{0.701}{1}\right)^2 6\,\frac{0.197}{1} = 8.71 = \boxed{9.40\ dB}$

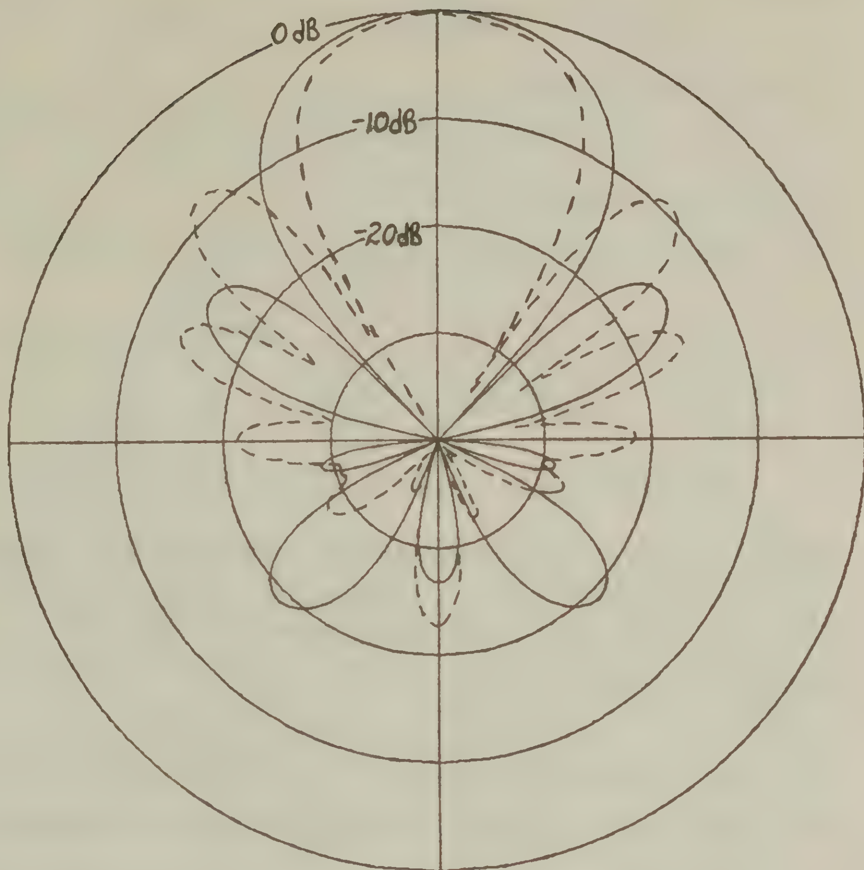
@ 520 MHz $\lambda = 0.577\,m$ $G = 15\left(\frac{0.701}{0.577}\right)^2 6\,\frac{0.197}{0.577} = 45.35 = \boxed{16.6\ dB}$

6.1-4 Helix with $N=12$ $C=0.197m$ $\alpha=8.53°$ $f=1525\,MHz$

$\lambda = 19.7\,cm$

$C/\lambda = \frac{19.7}{19.7} = 1$

$S/\lambda = \frac{C}{\lambda}\tan\alpha$
$= 1 \tan 8.53°$
$= 0.15$



— Calculated pattern using (6-22)

--- Measured pattern

6.2-1

Infinite biconical antenna impedance

θ_h	From (6-41) $Z_{in} = 120 \ln(\cot\frac{\theta_h}{2})$	From (6-42) $Z_{in} \approx 120 \ln(\frac{2}{\theta_h})$
0.1°	845.3 + j0 ohms	845.3 + j0 ohms
1°	568.96	568.96
10°	292.3	292.6
20°	208.2	209.5
50°	91.5	99.5 (above 20° limit on (6-42))

6.2-2

Finite biconical antenna $\theta_h = 1°$ cone lengths $= 0.3\lambda = h$

$Z_0 = 120 \ln(\cot\frac{\theta_h}{2}) = 120 \ln(\cot\frac{1°}{2}) = 569\ \text{ohms} \approx 600\ \text{ohms}$

From Fig. 6-10

$\boxed{Z_{in} = 170 + j380\ \text{ohms}}$ for $h = 0.3\lambda$

6.4-1 Equiangular spiral over 450 to 900 MHz

Use

$\epsilon = 4 \quad \Rightarrow \quad a = 0.221$

Then

$r = r_o e^{a\phi}$ for one spiral edge

Minimum radius

$$r_{min} = r(\phi=0) = r_o = \frac{\lambda_u}{4} = \frac{1}{4}\,\frac{3\times10^8}{900\times10^6} = \frac{1}{4}\,\frac{1}{3} = 0.083\ m = \boxed{8.3\,cm}$$

Maximum radius

$$r_{max} = \frac{1}{4}\lambda_L = \frac{1}{4}\,\frac{3\times10^8}{4.5\times10^8} = \frac{1}{4}\,\frac{2}{3} = 0.167\,m = \boxed{16.7\,cm}$$

$$= r_o e^{a\phi_m} = 8.3\, e^{0.221\phi_m}$$

so $e^{0.221\phi_m} = \frac{16.7}{8.3} \Rightarrow \phi_m = \frac{1}{0.221}\ln 2 = 3.13\,r \cong \pi = 180°$

$\phi_m = 180° \Rightarrow$ one-half turn on spiral

6.5-1 Self-complementary log-periodic toothed planar antenna

HP = 70° 400 MHz to 2 GHz

From Fig. 6-23 with HP = 70° $\tau = 0.81$

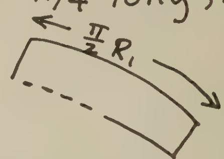
From (6-59)

$$\sigma = \sqrt{\tau} = \sqrt{0.81} = 0.9$$

For self-complementary $\alpha = 135°$ and $\beta = 45°$ in Fig. 6-22

At lowest frequency largest tooth should be $\lambda/4$ long, so

$\frac{\lambda_L}{4} = \frac{\pi}{2}R_1$ or $R_1 = \frac{2}{\pi}\frac{\lambda_L}{4}$ and $\lambda_L = \frac{300}{400} = \frac{3}{4}\,m$



$$R_1 = \frac{2}{\pi}\,\frac{3}{4} = \boxed{0.4775\,m}$$

At highest frequency shortest tooth should be about $\frac{\lambda_u}{4}$

so $R_u = \frac{2}{\pi}\frac{\lambda_u}{4}$ and $\lambda_u = \frac{300}{2000} = 0.15\,m$

$$R_u = \frac{2}{\pi}\,\frac{0.15}{4} = \boxed{0.0239\,m} = 2.39\,cm$$

From (6-52) and (6-53)

$$R_{n+1} = \tau R_n = 0.81\,R_n \qquad a_n = \sigma R_n = 0.9\,R_n$$

So

n	R_n	a_n	n	R_n	a_n
1	47.75 cm	42.975 cm	6	16.65	14.98
2	38.68	34.81	7	13.49	12.14
3	31.33	28.20	8	10.92	9.83
4	25.38	22.84	9	8.85	7.96
5	20.55	18.50	10	7.17	6.45
			11	5.81	5.22

6.5-1 (con't)

n	R_n	a_n
12	4.70	4.23
13	3.81	3.43
14	3.09	2.78
15	2.50	2.25
16	2.02	1.82

(See DuHamel & Isbell [15] for measurements on a very similar unit.)

6.5-2 LPDA   84 to 200 MHz $G = 9\,\text{dB}$

From Fig. 6-30 for optimum design and 9 dB gain

$$\tau = 0.865 \qquad \sigma = 0.158$$

The longest element is

$$L_1 = \frac{\lambda_L}{2} \text{ from (6-71)} \qquad \lambda_L = \frac{c}{f_L} = \frac{3\times10^8}{84\times10^6} = 3.571\,\text{m}$$

$$= 3.571/2 = 1.786\,\text{m}$$

At the upper frequency

$$L_N \sim \frac{\lambda_u}{2} = \frac{1}{2}\,\frac{3\times10^8}{200\times10^6} = \frac{1}{2}(1.5) = 0.75\,\text{m}$$

Other lengths and the spacings follow from (6-70) and (6-68)

$$L_{n+1} = \tau L_n = 0.865\,L_n \qquad d_n = 2\sigma L_n = 2(0.158)L_n = 0.316\,L_n$$

For example

$$L_2 = 0.865\,L_1 = 0.865(1.786) = 1.54\,\text{m}$$

$$d_1 = 0.316\,L_1 = 0.316(1.786) = 0.564\,\text{m}$$

Tabulating

n	L_n	d_n
1	1.786 m	0.564 m
2	1.545	0.488
3	1.336	0.422
4	1.156	0.365
5	1.000	0.316
6	0.865	0.273
7	0.748	

We stop at 7 elements since $L_7 = 0.748\,\text{m} < 0.75\,\text{m} = \frac{\lambda_u}{2}$.

6.5-3 LPDA of Example 6-3.

$$\tau = 0.917 \qquad \sigma = 0.169 \qquad L_1 = 0.75\,\text{m} \qquad L_{18} = 0.172\,\text{m}$$

n	$L_n = \tau L_{n-1}$	$d_n = 2\sigma L_n$
1	0.75 m	0.2535
2	0.688	0.232
3	0.631	0.213
4	0.578	0.195
5	0.530	0.179
6	0.486	0.164
7	0.446	0.151
8	0.409	0.138

6.5-3 (con't)

n	L_n	d_n	n	L_n	d_n
9	0.375 m	0.127 m	14	0.243	0.082
10	0.344	0.116	15	0.223	0.075
11	0.315	0.107	16	0.205	0.069
12	0.289	0.098	17	0.187	0.063
13	0.265	0.090	18	0.172	

6.5-4 LPDA

From Fig. 6-26 for optimum design

$$\tau = 0.917 \qquad \sigma = 0.17$$

The UHF band spans 470 (Ch.14) to 890 MHz (Ch.83)

Leaving one extra long element, the

$$L_2 = \frac{\lambda_L}{2} = \frac{1}{2}\frac{3\times10^8}{4.7\times10^8} = \frac{1}{2}(0.6383\,m) = 31.9\text{ cm}$$

The next to shortest (leaving room for one extra element) should be at least as short as

$$\frac{\lambda_u}{2} = \frac{1}{2}\frac{3\times10^8}{8.9\times10^8} = \frac{1}{2}(0.337\,m) = 16.9\,cm$$

First $L_1 = \frac{L_2}{\tau} = \frac{31.9\text{ cm}}{0.917} = 34.8\text{ cm}$

So

n	$L_n = \tau L_{n-1}$	$d_n = 2\sigma L_n$	
1	34.8 cm	11.8 cm	extra
2	31.9	10.8	
3	29.3	9.95	
4	26.8	9.12	
5	24.6	8.36	
6	22.5	7.67	
7	20.7	7.03	
8	19.0	6.45	
9	17.4	5.91	
10	15.9	5.42	
11	14.6		extra

CHAPTER 7

7.1-1

$$\psi(z,z') = \frac{e^{-j\beta R}}{4\pi R} = \frac{e^{-j\beta\sqrt{(z-z')^2+a^2}}}{4\pi\sqrt{(z-z')^2+a^2}}$$

$$\frac{\partial\psi(z,z')}{\partial z} = \frac{1}{4\pi}\, e^{-j\beta\sqrt{(z-z')^2+a^2}} \cdot \frac{\left(j\beta\sqrt{(z-z')^2+a^2}+1\right)(z-z')}{\left[(z-z')^2+a^2\right]^{3/2}}$$

$$\frac{\partial^2\psi(z,z')}{\partial z^2} = -\frac{e^{-j\beta R}}{4\pi R^5}\left[(z-z')^2\beta^2R^2 + j\beta R^3 + R^2 - 3(z-z')^2(j\beta R+1)\right]$$

$$= \frac{e^{-j\beta R}}{4\pi R^5}\left[-\beta^2R^4 + a^2\beta^2R^2 + (2R^2-3a^2)(1+j\beta R)\right]$$

$$\beta^2\psi(z,z') = \frac{e^{-j\beta R}}{4\pi R^5}\beta^2R^4$$

$$\therefore\ \frac{\partial^2\psi(z,z')}{\partial z^2} + \beta^2\psi(z,z') = \frac{e^{-j\beta R}}{4\pi R^5}\left[(1+j\beta R)(2R^2-3a^2) + a^2\beta^2R^2\right]$$

$$\therefore\ E_z^s = \frac{1}{4\pi j\omega\epsilon_0}\int_{-L/2}^{L/2} I(z')\,\frac{e^{-j\beta R}}{R^5}\left[(1+j\beta R)(2R^2-3a^2) + \beta^2a^2R^2\right]dz'$$

7.1-2

$$E_z^s = \frac{1}{j\omega\epsilon_0}\int I(z')\,\frac{\partial^2\psi(z,z')}{\partial z^2}\,dz' + \frac{\beta^2}{j\omega\epsilon_0}\int I(z')\,\psi(z,z')\,dz'$$

$$\frac{\beta^2}{j\omega\epsilon_0} = -j\omega\mu_0 \quad , \quad \psi(z,z') = \frac{e^{-j\beta R}}{4\pi R}$$

$$R = \sqrt{a^2+(z-z')^2}$$

1ST TERM:

$$\int I(z')\,\frac{\partial^2\psi(z,z')}{\partial z^2}\,dz' = -\int I(z')\,\frac{\partial}{\partial z}\,\frac{\partial\psi(z,z')}{\partial z'}\,dz'$$

$$= -\int I(z')\,\frac{\partial}{\partial z'}\left(\frac{\partial}{\partial z}\psi(z,z')\right)dz'$$

SINCE $\frac{\partial\psi}{\partial z'} = -\frac{\partial\psi}{\partial z}$.

$$\int_{-L/2}^{L/2} I(z')\,\frac{\partial^2\psi(z,z')}{\partial z^2}\,dz' = \underbrace{-\,I(z')\frac{\partial}{\partial z}\psi(z,z')\Big|_{-L/2}^{L/2}}_{\substack{\text{GOES TO ZERO}\\ \text{SINCE } I(\frac{L}{2})=0=I(-\frac{L}{2})}} + \int_{-L/2}^{L/2}\frac{\partial I(z')}{\partial z'}\frac{\partial\psi}{\partial z}\,dz'$$

$$\therefore\; E_z^s = -\int_{-L/2}^{L/2}\left[j\omega\mu_0 I(z') - \frac{1}{j\omega\epsilon_0}\,\frac{\partial I(z')}{\partial z'}\,\frac{\partial}{\partial z}\right]\frac{e^{-j\beta R}}{4\pi R}\,dz'$$

7.1-3

$$E^s = -j\omega\mu A_z - \frac{j}{\omega\epsilon}\,\frac{\partial^2 A_z}{\partial z^2} \qquad (1\text{-}89)$$

TANGENTIAL E-FIELD MUST VANISH ON THE SURFACE OF THE ANTENNA, SO

$$E_z^s = E^s + E^i = 0 \qquad (E^i \text{ IS THE INCIDENT FIELD})$$

$$\therefore\; \frac{\partial^2 A_z}{\partial z^2} + \beta^2 A_z = -j\omega\epsilon_0 E^i$$

A SOLUTION FOR THIS D.E. IS

$$A_z = C\sin\beta z + D\cos\beta z + A_p(z)$$

WHERE $A_p(z)$ IS A PARTICULAR SOLUTION AND $C\sin\beta z$ AND $D\cos\beta z$ ARE TWO INDEPENDENT HOMOGENEOUS SOLUTIONS.

IT CAN BE SHOWN THAT

$$A_p(z) = -\frac{j\omega\epsilon}{\beta}\int_{-L/2}^{L/2} E^i(\zeta)\,\sin\beta(z-\zeta)\,d\zeta$$

SO,

$$A_z = C\sin\beta z + D\cos\beta z - \frac{j\omega\epsilon}{\beta}\int_{-L/2}^{L/2} E^i(\zeta)\,\sin\beta(z-\zeta)\,d\zeta$$

THE DIPOLE HAS A SYMMETRIC CURRENT $\Rightarrow A_z$ SYMMETRIC

$$\therefore\; A_z = D\cos\beta z - \frac{j\omega\epsilon}{\beta}\int_{-L/2}^{L/2} E^i(\zeta)\,\sin\beta(z-\zeta)\,d\zeta$$

SUPPOSE INSTEAD OF AN INCIDENT FIELD $E^i(\zeta)$ WE HAVE A DELTA GAP VOLTAGE $V_T\,\delta(\zeta)$, THEN

$$-\frac{j\omega\epsilon}{\beta}\int_{-L/2}^{L/2} E^i(\zeta)\sin\beta(z-\zeta)\,d\zeta = -\frac{jV_T}{\eta}\int_{-L/2}^{L/2}\delta(\zeta)\sin\beta(z-\zeta)\,d\zeta$$

$$= -\frac{jV_T}{2\eta}\sin\beta|z|$$

$$\therefore A_z = D\cos\beta z - \frac{jV_T}{2\eta}\sin\beta|z|$$

$$= -\frac{j}{\eta}\left(C_1\cos\beta z + C_2\sin\beta|z|\right), \quad C_1 = -\frac{\eta}{j}D$$

$$C_2 = \frac{V_T}{2}$$

WE ALSO KNOW THAT

$$A_z = \int_{-L/2}^{L/2} I(z')\frac{e^{-j\beta R}}{4\pi R}\,dz', \quad R = \sqrt{(z-z')^2 + a^2}$$

$$\therefore \int_{-L/2}^{L/2} I(z')\frac{e^{-j\beta R}}{4\pi R}\,dz' = -\frac{j}{\eta}\left(C_1\cos\beta z + \frac{V_T}{2}\sin\beta|z|\right)$$

7.2-1

IMPEDANCE MATRIX VALUES ARE GIVEN ON THE NEXT PAGE FOR A DIPOLE WITH:

$a = 0.005\lambda$
$L = 0.47\lambda$
$N = 56$

THE STUDENT SHOULD BE TOLD THAT INCREASED SAMPLING IN THE NUMERICAL INTEGRATION SCHEME SHOULD BE USED NEAR THE SEGMENT WHERE $R \to a$. THE DATA ON THE NEXT PAGE USED INCREASED SAMPLING FOR Z_{11}, Z_{12}, Z_{13}, Z_{14}. OTHERWISE 5 SAMPLES WERE USED.

Z_{1n}

$n=1,\cdots,56$

$\boxed{\text{Part (a)}}$

1	-.662542930463D+01	.143979738379D+06
2	-.662358698260D+01	-.480333098163D+05
3	-.661806224804D+01	-.142855703355D+05
4	-.660886166099D+01	-.474178350210D+04
5	-.659614789078D+01	-.202854815624D+04
6	-.657963202841D+01	-.107607955122D+04
7	-.655948607865D+01	-.639732314500D+03
8	-.653573400326D+01	-.412591442789D+03
9	-.650840403733D+01	-.282813436426D+03
10	-.647752863744D+01	-.203225216416D+03
11	-.644314444300D+01	-.151624142143D+03
12	-.640529220932D+01	-.116631464661D+03
13	-.636401674651D+01	-.920053084771D+02
14	-.631936685630D+01	-.741260572723D+02
15	-.627139525057D+01	-.607953701401D+02
16	-.622015847434D+01	-.506248576084D+02
17	-.616571681731D+01	-.427077860301D+02
18	-.610813422273D+01	-.364345366684D+02
19	-.604747818779D+01	-.313847012388D+02
20	-.598381966195D+01	-.272617814737D+02
21	-.591723293805D+01	-.238524574727D+02
22	-.584779553924D+01	-.210004899747D+02
23	-.577558810109D+01	-.185895932683D+02
24	-.570069424848D+01	-.165319504452D+02
25	-.562320046959D+01	-.147603607565D+02
26	-.554319598467D+01	-.132227745836D+02
27	-.546077261076D+01	-.118784281853D+02
28	-.537602462436D+01	-.106950691879D+02
29	-.528904861869D+01	-.964693774684D+01
30	-.519994335886D+01	-.871327901154D+01
31	-.510880963463D+01	-.787723425132D+01
32	-.501575010895D+01	-.712500528906D+01
33	-.492086916603D+01	-.644521851776D+01
34	-.482427275579D+01	-.582843626860D+01
35	-.472606823738D+01	-.526677808182D+01
36	-.462636422039D+01	-.475362473423D+01
37	-.452527040519D+01	-.42833[illegible]514943D+01
38	-.442289742283D+01	-.385131148854D+01
39	-.431935667246D+01	-.345335145159D+01
40	-.421476016018D+01	-.308602953961D+01
41	-.410922033639D+01	-.274635100759D+01
42	-.400284993383D+01	-.243172372570D+01
43	-.389576180578D+01	-.213989425559D+01
44	-.378806876460D+01	-.18688[illegible]528602D+01
45	-.367988342155D+01	-.161700219300D+01
46	-.357131802804D+01	-.138269697305D+01
47	-.346248431701D+01	-.116463816330D+01
48	-.335349334728D+01	-.961635649320D+00
49	-.324445534957D+01	-.772629480037D+00
50	-.313547957394D+01	-.596671985600D+00
51	-.302667414101D+01	-.432912627342D+00
52	-.291814589402D+01	-.28058[illegible]118322D+00
53	-.281000025560D+01	-.138996438929D+00
54	-.270234108672D+01	-.751743971647D-02
55	-.259527054887D+01	.114426222342D+00
56	-.248888897061D+01	.227356020677D+00

Part (b)

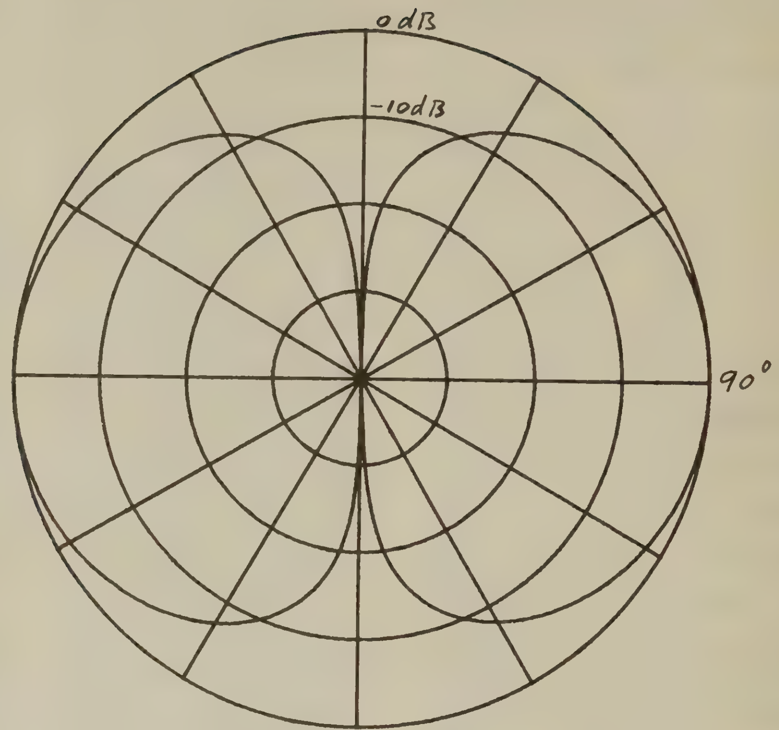
VALUES FOR (I_n), $n=1,\dots,56$

NOTE: STUDENT SHOULD CHECK VALUES FOR (V_m) AGAINST TABLE 7-1 BEFORE COMPUTING (V_m), $m=1,\dots,56$ AND (I_n).

1	-.122426948798D-02	-.371609428441D-03
2	-.195572977199D-02	-.603694918286D-03
3	-.262391845053D-02	-.822845259143D-03
4	-.324668569612D-02	-.103412029596D-02
5	-.383565852645D-02	-.124082949861D-02
6	-.439679423598D-02	-.144465305950D-02
7	-.493312037866D-02	-.164643014274D-02
8	-.544627789082D-02	-.184660540409D-02
9	-.593711954169D-02	-.204540109847D-02
10	-.640602221662D-02	-.224290727255D-02
11	-.685306236756D-02	-.243913323799D-02
12	-.727812180156D-02	-.263403978070D-02
13	-.768095523940D-02	-.282756130043D-02
14	-.806123550813D-02	-.301962267050D-02
15	-.841858496977D-02	-.321015365811D-02
16	-.875259811533D-02	-.339910289463D-02
17	-.906285826539D-02	-.358645315637D-02
18	-.934895021691D-02	-.377223995813D-02
19	-.961047001555D-02	-.395657626256D-02
20	-.984703262883D-02	-.413968780696D-02
21	-.100582780525D-01	-.432196700810D-02
22	-.102438762128D-01	-.450406068668D-02
23	-.104035309176D-01	-.468702315086D-02
24	-.105369830439D-01	-.487260638237D-02
25	-.106440130910D-01	-.506388976563D-02
26	-.107244431911D-01	-.526654177346D-02
27	-.107781386514D-01	-.549298717694D-02
28	-.108050090717D-01	-.579506463014D-02
29	-.108050090717D-01	-.579506463020D-02
30	-.107781386515D-01	-.549298717703D-02
31	-.107244431912D-01	-.526654177360D-02
32	-.106440130909D-01	-.506388976566D-02
33	-.105369830441D-01	-.487260638243D-02
34	-.104035309176D-01	-.466702315092D-02
35	-.102438762129D-01	-.450406068671D-02
36	-.100582780523D-01	-.432196700816D-02
37	-.984703262844D-02	-.413968780708D-02
38	-.961047001544D-02	-.395657626268D-02
39	-.934895021725D-02	-.377223995813D-02
40	-.906285826550D-02	-.358645315642D-02
41	-.875259811544D-02	-.339910289467D-02
42	-.841858496971D-02	-.321015365815D-02
43	-.806123550813D-02	-.301962267052D-02
44	-.768095523943D-02	-.282756130047D-02
45	-.727812180153D-02	-.263403978068D-02
46	-.685306236749D-02	-.243913323798D-02
47	-.640602221665D-02	-.224290727263D-02
48	-.593711954164D-02	-.204540109847D-02
49	-.544627789073D-02	-.184660540411D-02
50	-.493312037866D-02	-.164643014274D-02
51	-.439679423598D-02	-.144465305952D-02
52	-.383565852641D-02	-.124082949864D-02
53	-.324668569609D-02	-.103412029602D-02
54	-.262391845056D-02	-.822845259147D-03
55	-.195572977202D-02	-.603694918282D-03
56	-.122426948800D-02	-.371609428446D-03

Part (c)

DB PLOT



Part (d)

$$(V_m) = \begin{pmatrix} 1.0 + j\,0.0 \\ 1.0 + j\,0.0 \\ \vdots \\ 1.0 + j\,0.0 \end{pmatrix} \quad ; \text{ SEE PG. 335}$$

7.2-2 CONSIDER EQ. IN PROB 7.1-1 DIVIDED BY $I(z')$ WHOSE UNITS ARE THEN

$$\frac{1}{\frac{\text{FARAD}}{\text{SEC-m}}}\left[\frac{m^3}{m^5}\right] = \frac{\text{SEC}-m^4}{\text{FARAD}-m^5} = \frac{\text{SEC}}{\text{FARAD-m}}$$

FROM KRAUS & CARVER APPENDIX A, DIMENSION OF FARAD $= \dfrac{I^2T^4}{ML^2}$

$$\therefore \frac{\text{SEC}}{\text{FARAD-m}} \Rightarrow \frac{TML^2}{I^2T^4L} = \frac{ML}{I^2T^3}\cdot\frac{L}{L} = \text{OHM}/m$$

IF (V_m) IS IN VOLTS/m WITH $[Z_{mn}]$ IN OHM/m, THEN (I_n) IS IN AMPS.

7.3-1

Z_{1n} for

$n = 1, \cdots, 56$

with

$a = 0.005\lambda$

$L = 0.47\lambda$

$N = 56$

1	-.662542909639D+01	.143941251929D+06
2	-.662358677131D+01	-.480119383707D+05
3	-.661806203584D+01	-.142872428594D+05
4	-.660886144908D+01	-.474210263681D+04
5	-.659599597394D+01	-.20858[illegible]500118D+04
6	-.657948093107D+01	-.109593386503D+04
7	-.655933597811D+01	-.647965704288D+03
8	-.653558507797D+01	-.416478568799D+03
9	-.650825646193D+01	-.284836431447D+03
10	-.647738258840D+01	-.204360810383D+03
11	-.644300009059D+01	-.152301279997D+03
12	-.640514972162D+01	-.117055631043D+03
13	-.636387629047D+01	-.922821274884D+02
14	-.631922859407D+01	-.743130798409D+02
15	-.627125934208D+01	-.609255278087D+02
16	-.622002507536D+01	-.507177975876D+02
17	-.616558608011D+01	-.427756597234D+02
18	-.610800629501D+01	-.364850990048D+02
19	-.604735321333D+01	-.31423[illegible]397877D+02
20	-.598369777997D+01	-.272913165045D+02
21	-.591711428287D+01	-.238755390783D+02
22	-.584768024066D+01	-.21018[illegible]648688D+02
23	-.577547628324D+01	-.186042358973D+02
24	-.570058603066D+01	-.16543[illegible]119845D+02
25	-.562309596556D+01	-.147700671850D+02
26	-.554309530244D+01	-.13230[illegible]924699D+02
27	-.546067585283D+01	-.118851095879D+02
28	-.537593188715D+01	-.107006827872D+02
29	-.528895999249D+01	-.965169075335D+01
30	-.519985892806D+01	-.871733278833D+01
31	-.510872947709D+01	-.788071559934D+01
32	-.501567429641D+01	-.712801469047D+01
33	-.492079776362D+01	-.644783622698D+01
34	-.482420582225D+01	-.583072685581D+01
35	-.472600582460D+01	-.526879385654D+01
36	-.462630637388D+01	-.475540834755D+01
37	-.452521716372D+01	-.428497158852D+01
38	-.442284881827D+01	-.385272961728D+01
39	-.431931273010D+01	-.345462521038D+01
40	-.421472089851D+01	-.308717888500D+01
41	-.410918576715D+01	-.27473[illegible]265518D+01
42	-.400282006210D+01	-.243267173846D+01
43	-.389573662976D+01	-.214076051562D+01
44	-.378804827596D+01	-.186968987244D+01
45	-.367986760540D+01	-.161773368785D+01
46	-.357130686267D+01	-.138337270972D+01
47	-.346247777434D+01	-.116526442923D+01
48	-.335349139288D+01	-.962217852178D+00
49	-.324445794269D+01	-.773172285058D+00
50	-.313548666758D+01	-.597179428772D+00
51	-.302668568190D+01	-.433388209652D+00
52	-.291816182303D+01	-.281031884493D+00
53	-.281002050773D+01	-.139417043392D+00
54	-.270236559120D+01	-.791420180195D-02
55	-.259529922952D+01	.114050271444D+00
56	-.248892174563D+01	.227001098409D+00

7.3-1 (CONT.)

I_n for $n = 1, \cdots, 56$

-.135130395448D-02	.418898812782D-03
-.216366763805D-02	.659944835067D-03
-.290922366509D-02	.873573315949D-03
-.360735956401D-02	.106618758118D-02
-.427100102524D-02	.124204038115D-02
-.490597553655D-02	.140308440811D-02
-.551522322291D-02	.155033138874D-02
-.610020653380D-02	.168431019309D-02
-.666155738562D-02	.180527754030D-02
-.719940939955D-02	.191332585841D-02
-.771358506477D-02	.200844325992D-02
-.820370894888D-02	.209054865740D-02
-.866928020991D-02	.215951270433D-02
-.910972129639D-02	.221516983992D-02
-.952441198296D-02	.225732406488D-02
-.991271398572D-02	.228574959788D-02
-.102739892933D-01	.230018654764D-02
-.106076141604D-01	.230033079637D-02
-.109129900183D-01	.228581610578D-02
-.111895521245D-01	.225618454365D-02
-.114367765109D-01	.221083781099D-02
-.116541856173D-01	.214895486310D-02
-.118413528728D-01	.206934516521D-02
-.119979064236D-01	.197016709786D-02
-.121235321414D-01	.184832916735D-02
-.122179760120D-01	.169799752926D-02
-.122810459725D-01	.150650853733D-02
-.123126132534D-01	.122189034429D-02
-.123126132535D-01	.122189034416D-02
-.122810459725D-01	.150650853720D-02
-.122179760118D-01	.169799752917D-02
-.121235321415D-01	.184832916728D-02
-.119979064234D-01	.197016709779D-02
-.118413528729D-01	.206934516513D-02
-.116541856176D-01	.214895486304D-02
-.114367765110D-01	.221083781092D-02
-.111895521243D-01	.225618454359D-02
-.109129900183D-01	.228581610566D-02
-.106076141604D-01	.230033079630D-02
-.102739892934D-01	.230018654755D-02
-.991271398583D-02	.228574959784D-02
-.952441198290D-02	.225732406479D-02
-.910972129645D-02	.221516983992D-02
-.866928020997D-02	.215951270433D-02
-.820370894888D-02	.209054865734D-02
-.771358506486D-02	.200844325991D-02
-.719940939955D-02	.191332585840D-02
-.666155738562D-02	.180527754029D-02
-.610020653382D-02	.168431019306D-02
-.551522322294D-02	.155033138872D-02
-.490597553667D-02	.140308440811D-02
-.427100102532D-02	.124204038115D-02
-.360735956398D-02	.106618758115D-02
-.290922366509D-02	.873573315938D-03
-.216366763807D-02	.659944835046D-03
-.135130395447D-02	.418898812780D-03

7.4-1

$$Z_{mn} = \int_{-\frac{l_m}{2}}^{\frac{l_m}{2}} \bar{W}_m(l_m) \cdot \bar{E}_n^s(l_m)\, dl$$

Let the test function have the same form as the current:

$$Z_{mn} = \int_{-\frac{l_m}{2}}^{\frac{l_m}{2}} \bar{I}_m(l_m) \cdot \bar{E}_n^s(l_m)\, dl$$

Let the integration be over a surface:

$$Z_{mn} = \iint_S \bar{J}_m \cdot \bar{E}_n^s\, ds$$

If magnetic sources are present, then from (1-159)

$$Z_{mn} = \iint_S (\bar{J}_m \cdot \bar{E}_n^s - \bar{M}_m \cdot \bar{H}_n^s)\, ds$$

Argue similarly for V_m.

7.5-1

$$Z_{mn} = \iint_S (\bar{J}_m \cdot \bar{E}_n^s - \bar{M}_m \cdot \bar{H}_n^s)\, ds$$

Here $\bar{M}_m = 0$ and we have only a current filament

So, $Z_{mn} = \int \bar{I}_m \cdot \bar{E}_n^s\, dl$, but $\bar{I}_m \longrightarrow \bar{W}_m$,

$$Z_{mn} = \int \bar{W}_m \cdot \bar{E}_n^s\, dl = \int \bar{E}_m^s \cdot \bar{W}_n\, dl$$

by reciprocity.

$$\therefore Z_{mn} = \int_{z_{m-1}}^{z_m} \frac{\sin\beta(z - z_{m-1})}{\sin(\beta \Delta z_m)}\, \hat{z} \cdot \bar{E}_n^s\, dz + \int_{z_m}^{z_{m+1}} \frac{\sin\beta(z_{m+1} - z)}{\sin(\beta \Delta z_m)}\, \hat{z} \cdot \bar{E}_n^s\, dz$$

7.5-2

$$E_z = \frac{1}{j\omega\epsilon_0}\int_{z_1}^{z_2} I(z')\left[\frac{\partial^2}{\partial z^2}\frac{e^{-j\beta R}}{4\pi R} + \beta^2\frac{e^{-j\beta R}}{4\pi R}\right]dz'$$

NOTE: $\frac{\partial\psi}{\partial z} = -\frac{\partial\psi}{\partial z'}$; $\frac{\partial^2\psi}{\partial z^2} = \frac{\partial^2\psi}{\partial z'^2}$

We integrate the first term twice by parts.

$$\int_{z_1}^{z_2}\frac{\partial^2\psi}{\partial z^2}I(z')\,dz' = \int_{z_1}^{z_2}\frac{\partial^2\psi}{\partial z'^2}I(z')dz' = \int_{z_1}^{z_2} I(z')\,d\left(\frac{\partial\psi}{\partial z'}\right)$$

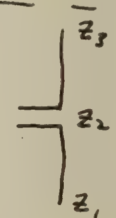
$$= I(z')\frac{\partial\psi}{\partial z'}\Big|_{z_1}^{z_2} - \int_{z_1}^{z_2}\frac{dI(z')}{dz'}\frac{\partial\psi}{\partial z'}dz'$$

$$= I(z')\frac{\partial\psi}{\partial z'}\Big|_{z_1}^{z_2} - \frac{dI(z')}{dz'}\psi\Big|_{z_1}^{z_2} + \int_{z_1}^{z_2}\frac{d^2I(z')}{dz'^2}\psi\,dz'$$

$$\therefore\ E_z^1 = \frac{j}{\omega\epsilon_0}\left[\frac{dI(z')}{dz'}\psi(z,z') + I(z')\frac{\partial\psi(z,z')}{\partial z}\right]_{z'=z_1}^{z'=z_2}$$

$$+ \frac{1}{j\omega\epsilon_0}\int_{z_1}^{z_2}\left[\frac{d^2I(z')}{dz'^2} + \beta^2 I(z')\right]\psi(z,z')\,dz'$$

7.5-3



$$I(z')\begin{cases} = \dfrac{\sin\beta(z'-z_1)}{\sin\beta(z_2-z_1)}, & z_1 \le z' < z_2 \\[2ex] = \dfrac{\sin\beta(z_3-z')}{\sin\beta(z_3-z_2)}, & z_2 \le z' < z_3 \end{cases}$$

$$I'(z'\begin{cases} = \dfrac{\beta\cos\beta(z'-z_1)}{\sin\beta(z_2-z_1)}, & z_1 \le z' < z_2 \\[2ex] = \dfrac{-\beta\cos\beta(z_3-z')}{\sin\beta(z_3-z_2)}, & z_2 \le z' < z_3 \end{cases}$$

$$I''(z')\begin{cases} = \dfrac{-\beta^2\sin\beta(z'-z_2)}{\sin\beta(z_2-z_1)}, & z_1 \le z' < z_2 \\[2ex] = \dfrac{-\beta^2\sin\beta(z_3-z')}{\sin\beta(z_3-z_2)}, & z_2 \le z' \le z_3 \end{cases}$$

7.5-3 (CONT.)

We note that the last term (integral term) in (7-55) for E_z vanishes.

$$\therefore\ E_z = \frac{j}{\omega\epsilon_0}\left[I'(z')\psi + I(z')\frac{\partial\psi}{\partial z}\right]_{z_1}^{z_3}$$

$$\frac{\partial\psi}{\partial z} = \frac{\partial}{\partial z}\left[\frac{e^{-j\beta R}}{4\pi R}\right] = \frac{\partial}{\partial R}\left[\frac{e^{-j\beta R}}{4\pi R}\right]\frac{dR}{dz} = \frac{1}{4\pi}\left[\frac{-j\beta}{R} - \frac{1}{R^2}\right]e^{-j\beta R}\cdot\frac{(z-z')}{R}$$

$$E_z = \frac{j}{\omega\epsilon_0}\left[\frac{\beta\cos\beta(z'-z_1)}{\sin\beta(z_2-z_1)}\frac{e^{-j\beta R}}{4\pi R} + \frac{\sin\beta(z'-z_1)}{\sin\beta(z_2-z_1)}\left[\frac{-j\beta}{R} - \frac{1}{R^2}\right]\cdot\frac{e^{-j\beta R}}{4\pi R}(z-z')\right]_{z_1}^{z_2}$$

$$+\frac{j}{\omega\epsilon_0}\left[\frac{-\beta\cos\beta(z_3-z')}{\sin\beta(z_3-z_2)}\frac{e^{-j\beta R}}{4\pi R} + \frac{\sin\beta(z_3-z')}{\sin\beta(z_3-z_2)}\left[\frac{-j\beta}{R} - \frac{1}{R^2}\right]\cdot\frac{e^{-j\beta R}}{4\pi R}(z-z')\right]_{z_2}^{z_3}$$

After some algebraic manipulation

$$E_z = \frac{j}{4\pi\omega\epsilon_0}\left\{\frac{-\beta e^{-j\beta R_1}}{R_1\sin\beta(z_2-z_1)} - \frac{\beta e^{-j\beta R_3}}{R_3\sin\beta(z_3-z_2)} + \beta\underbrace{\left[\frac{\cos\beta(z_2-z_1)}{\sin\beta(z_2-z_1)} + \frac{\cos\beta(z_3-z_2)}{\sin\beta(z_3-z_2)}\right]}_{\Downarrow}\frac{e^{-j\beta R_2}}{R_2}\right.$$

$$\boxed{\frac{\beta}{4\pi\omega\epsilon_0} = 30}$$

$$\frac{\sin\beta(z_3-z_1)}{\sin\beta(z_2-z_1)\sin\beta(z_3-z_2)}$$

$$\therefore E_z = j30\left\{-\frac{e^{-j\beta R_1}}{R_1\sin\beta(z_2-z_1)} - \frac{e^{-j\beta R_3}}{R_3\sin\beta(z_3-z_2)} + \frac{\sin\beta(z_3-z_1)}{\sin\beta(z_2-z_1)\sin\beta(z_3-z_2)}\frac{e^{-j\beta R_2}}{R_2}\right\}$$

Generalize to obtain (7-59)

7.5-4

$$\bar{E} = -\frac{1}{j\omega\epsilon_0}\nabla\nabla\cdot\bar{A} - j\omega\mu_0\bar{A} \quad , \quad \bar{A} = \hat{z}A_z$$

$$\therefore E_\rho = -\frac{1}{j\omega\epsilon_0}\frac{\partial}{\partial\rho}\frac{\partial A_z}{\partial z}$$

$$E_\rho = -\frac{1}{j\omega\epsilon_0}\int\frac{\partial}{\partial\rho}\frac{\partial}{\partial z}\left[G(z,z')\,I(z')\right]dz'$$

$$= \frac{-1}{j\omega\epsilon_0}\int\frac{\partial}{\partial\rho}\Big[I(z')\frac{\partial}{\partial z}G(z,z') + G(z,z')\underbrace{\frac{\partial}{\partial z}I(z')}_{\text{ZERO}}\Big]dz'$$

$$= \frac{-1}{j\omega\epsilon_0}\left\{\int I(z')\frac{\partial^2}{\partial\rho\,\partial z}G(z,z')\,dz' + \frac{\partial I(z')}{\partial z'}\frac{\partial z'}{\partial\rho}G(z,z')\Big|_{z_1}^{z_2}\right.$$

$$\left. - \int I(z')\frac{\partial}{\partial\rho}\frac{\partial}{\partial z}G(z,z')\,dz'\right\}$$

$$= \frac{-1}{j\omega\epsilon_0}\, I'(z')\,G(z,z')\cot\theta_i\Big|_{z_1}^{z_2}$$

$$= \frac{1}{j\omega\epsilon_0 4\pi}\, I'(z')\,\frac{e^{-j\beta R}}{R}\,\frac{\cos\theta_i}{\sin\theta_i}\Big|_{z_1}^{z_2}$$

$$= \frac{1}{j\omega\epsilon_0 4\pi\rho}\, I'(z')\,e^{-j\beta R}\cos\theta_i\Big|_{z_1}^{z_2}$$

$$\therefore E_\rho = \frac{-j}{4\pi\omega\epsilon_0\rho}\sum_{i=1}^{N}\Delta I_i'\,e^{-j\beta R_i}\cos\theta_i$$

$$\Delta I_i' = \lim_{\epsilon\to 0}\left[I'(z_i-\epsilon) - I'(z_i+\epsilon)\right]$$

For the PWS

$$E_\rho = \frac{-j}{4\pi\omega\epsilon_0\rho}\sum_{i=1}^{3}\Delta I_i'\,e^{-j\beta R_i}\cos\theta_i$$

$$= \frac{-j}{4\pi\omega\epsilon_0\rho}\Big[\beta\,\frac{\cos\beta\overbrace{(z_{n-1}-z_{n-1})}^{\text{ZERO}}}{\sin\beta(z_n - z_{n-1})}e^{-j\beta R_{n-1}}\cos\theta_1$$

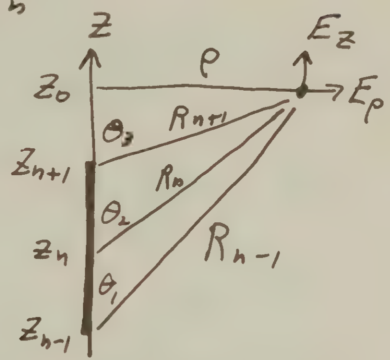
$$-\beta\,\frac{\cos\beta(z_n - z_{n-1})}{\sin\beta(z_n - z_{n-1})}e^{-j\beta R_n}\cos\theta_2$$

$$-\beta\,\frac{\cos\beta(z_{n+1} - z_n)}{\sin\beta(z_{n+1} - z_n}e^{-j\beta R_n}\cos\theta_2$$

$$+\beta\,\frac{\cos\beta(z_{n+1} - z_{n+1}}{\sin\beta(z_{n+1} - z_n)}e^{-j\beta R_{n+1}}\cos\theta_3\Big]$$

7.5-4 (CONT.)

$$\therefore E_\rho = \frac{j\beta}{4\pi\rho \sin(\beta\Delta z_n)}\left[(z_{n-1}-z_0)\frac{e^{-j\beta R_{n-1}}}{R_{n-1}} - 2(z_n - z_0)\cos(\beta\Delta z_n)\frac{e^{-j\beta R_n}}{R_n} + (z_{n+1}-z_0)\frac{e^{-j\beta R_{n+1}}}{R_{n+1}}\right]$$



7.6-1

$$\bar{E} = j\omega\mu_0 A_z \sin\theta\ \hat{\theta}$$

$$A_z = \frac{1}{4\pi}\int I(z')\frac{e^{-j\beta R}}{R}dz' \quad , \quad R \approx r - z'\cos\theta$$

$$A_z = \frac{e^{-j\beta r}}{4\pi r}\int I(z')\,e^{j\beta z'\cos\theta}dz'$$

For the n^{th} expansion function or PWS mode:

$$A_{z_n} = \frac{e^{-j\beta r_0}}{4\pi r_0}\left[I_n\int_{z_{n-1}}^{z_n}\frac{\sin\beta(z'-z_{n-1})}{\sin\beta(z_n - z_{n-1})}e^{j\beta z'\cos\theta}dz' + I_n\int_{z_n}^{z_{n+1}}\frac{\sin\beta(z_{n+1}-z')}{\sin\beta(z_{n+1}-z_n)}\cdot e^{j\beta z'\cos\theta}\right]$$

For all N expansion functions with the assumption that all segments are the same Length,

$$A_z = \frac{e^{-j\beta r_0}}{4\pi r_0}\sum_{n=1}^{N}\frac{I_n}{\sin\beta(\Delta z_n)}\left[\int_{z_{n-1}}^{z_n}\sin\beta(z'-z_{n-1})\,e^{j\beta z'\cos\theta} + \int_{z_n}^{z_{n+1}}\sin\beta(z_{n+1}-z')e^{j\beta z'\cos\theta}dz'\right]$$

$$E_\theta(\theta) = j\omega\mu_0 \sin\theta\, A_z$$

7.6-2

$$P_{ave} = \frac{1}{2}\,\mathrm{Re}\left[\iint_S \bar{E}\times\bar{H}^* \cdot d\bar{s}\right]$$

$$= \frac{1}{2}\,\mathrm{Re}\left[\iint_S \left[E_\theta H_\phi^* - E_\phi H_\theta^*\right] r_0^2 \sin\theta\, d\theta\, d\phi\right]$$

$$= \frac{1}{2\eta}\iint \left[|E_\theta|^2 + |E_\phi|^2\right] r_0^2 \sin\theta\, d\theta\, d\phi$$

$$G = \frac{4\pi U_m}{P_{in}}$$

$$U(\theta,\phi) = \frac{1}{2}\,\mathrm{Re}(\bar{E}\times\bar{H}^*)\, r_0^2 = \frac{1}{2\eta}\left[|E_\theta|^2 + |E_\phi|^2\right] r_0^2$$

$$P_{in} = \frac{1}{2}|I_{in}|^2 R_{in}$$

$$\therefore\ G = \frac{4\pi}{2\eta}\,\frac{\left[|E_\theta|^2 + |E_\phi|^2\right] r_0^2}{\frac{1}{2}|I_n|^2 R_{in}}$$

$$G = \frac{\left[|E_\theta|^2 + |E_\phi|^2\right] r_0^2}{30\,|I_{in}|^2 R_{in}}$$

7.7-1

$$V_m = -\iint_S \left(\bar{J}_m\cdot\bar{E}^i - \bar{M}_m\cdot\bar{H}^i\right) ds \qquad ; \quad \bar{J}_m \equiv 0$$

$$V_m = \iint_S \bar{H}_m\cdot\bar{M}^i ds \qquad ; \quad \bar{H}_m\cdot\bar{M}^i = \frac{\delta(z)}{2\pi a}$$

$$V_m = \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}}\int_0^{2\pi} \frac{\delta(z)}{2\pi a}\, a\, d\phi\, dz = \int_{-\delta/2}^{\delta/2} \delta(z)\, dz \equiv 1$$

(Refer to fig. 7-17a.)

7.7-2

$$M_{\phi'} = \frac{-1}{\rho' \ln(b/a)} \qquad (\ \to 0)$$

When $\rho = 0$, we have $R = \left[(z-z')^2 + \rho'^2\right]^{1/2}$, so

$$F_\phi(0,z) = -\frac{\epsilon_0}{4\pi}\frac{1}{\ln(b/a)} \cdot \int_a^b \frac{e^{-j\beta R}}{R}\left[\int_0^{2\pi} \cos\phi'\, d\phi'\right] d\rho' \equiv 0$$

Then $H_\phi(0,z) \equiv 0$.

$$E_z = -\frac{1}{\epsilon_0}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\phi)$$

$$E_z(0,z) = -\frac{1}{\epsilon_0}\frac{F_\phi}{\rho} - \frac{1}{\epsilon_0}\frac{\partial}{\partial\rho}F_\phi = -\frac{2}{\epsilon_0}\frac{\partial F_\phi}{\partial\rho}\bigg|_{\rho=0}$$

where l'Hospital's rule is used to account for the $0/0$ term. Interchanging the order of integration and differentiation and carrying out the details, we find simply that

$$E_z(0,z) = \frac{1}{\pi}\frac{1}{\ln(b/a)} \cdot \int_a^b \int_0^\pi \cos\phi' \left[\frac{\partial}{\partial\rho}\frac{e^{-j\beta R}}{R}\right]_{\rho=0} d\phi' d\rho'$$

$$E_z(0,z) = \frac{1}{2\ln(b/a)}\left(\frac{e^{-j\beta\sqrt{z^2+a^2}}}{\sqrt{z^2+a^2}} - \frac{e^{-j\beta\sqrt{z^2+b^2}}}{\sqrt{z^2+b^2}}\right)$$

7.8-1

$$[S]_{ij} = [S]_{1,|i-j|+1}, \quad i \geq 2,\ j \geq 1. \qquad (7\text{-}88)$$

For the first row of submatrices we have

$$[S]_{11}\ [S]_{12}\ [S]_{13}\ \cdots\ [S]_{1j}$$

7.8-1

Using (7-88)

$$[S]_{21} = [S]_{1,|2-1|+1} = [S]_{12}$$

$$[S]_{31} = [S]_{1,|3-1|+1} = [S]_{13}$$

$$[S]_{j1} = [S]_{1,|j+1|+1} = [S]_{1j}$$

$$[S]_{22} = [S]_{1,|2-2|+1} = [S]_{11}$$

$$[S]_{23} = [S]_{1,|2-3|+1} = [S]_{12}$$

$$[S]_{2j} = [S]_{1,|2-j|+1} = [S]_{1(j-1)}$$

In general

$$[S]_{j1} = [S]_{1j}$$

$$[S]_{j2} = [S]_{1,|j-2|+1} = [S]_{1(j-1)}$$

$$[S]_{jj} = [S]_{1,(j-j)+1} = [S]_{11}$$

And (7-87) follows.

7.9-1

The extension to 2 or more LPDA's closely follows pgs. 345-349. Hence, the following equations in the text apply:

$$[Z_A]_{ij} = \frac{V_i}{I_j} \qquad (7\text{-}94)$$

$$[I_s] = \left[[Y_A] + [Y_T]\right][V_A] \qquad (7\text{-}95)$$

except that now there are at least 2 non-zero

7.9-1 (cont.)

values in $[I_s]$. i.e. $[I_s]$ becomes $[I_{s_p}]$

$$[I] = [Z_{mn}]^{-1}[V] = [Y_{mn}][V] \qquad (7\text{-}96)$$

or

$$I_m = \sum_{m=1}^{M\times N\times P} Y_{mn} V_i \quad , \quad m = 1, 2, \cdots M\times N\times P \qquad \begin{matrix}(7\text{-}97)\\ \text{(NEW)}\end{matrix}$$

where P = number of identical LPDAs.

$$I_j = \sum_{i=1}^{N\times P} Y_{ji} V_i \qquad j = 1, 2, \cdots N\times P \qquad \begin{matrix}(7\text{-}98)\\ \text{(NEW)}\end{matrix}$$

or

$$[I_A] = [Y_A][V_A] \qquad (7\text{-}99)$$

$[Y_T] = [\quad]$ as in (7-100), but for the array of 2 or more LPDA's

$$[Y_{T_P}] = \begin{bmatrix} [Y_T] & 0 & 0 & \cdots & 0 \\ 0 & [Y_T] & 0 & \cdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & [Y_T] \end{bmatrix}$$

So,

$$[V_A] = \left[[Y_A] + [Y_{T_P}]\right]^{-1}[I_{s_p}] \qquad \begin{matrix}(7\text{-}101)\\ \text{(NEW)}\end{matrix}$$

and

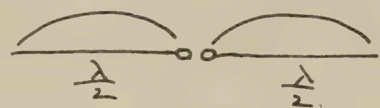
$$[I_n] = [Z_{mn}]^{-1}[V_m] \qquad (7\text{-}102)$$

where the order of $[I_n]$ is $M\times N\times P$.

7.9-2 (This is an important conceptual problem.)

a) No!!! Because $[Z_A]$ by method A is obtained without just considering two dipoles at a time as in method B. Theoretically method B will fail when some of the LPDA elements are 1λ, 2λ, etc. Long.

i.e.



NOTE: TWO $\lambda/2$ (RESONANT) SCATTERERS.

Method A will not have difficulty with this situation.

Method B assumes that the dipoles not under consideration in calculating an element of $[Z_A]$ absorb and reradiate negligible amounts of energy. Clearly this is not true if some of the dipoles have arms that are $\frac{\lambda}{2}$, $3\frac{\lambda}{2}$, etc.

b) For purposes of explanation, consider PWS expansion functions.

Suppose we think of each expansion function as being a dipole with a single PWS mode.

When we compute $[Z_{mn}]$ in the moment method, we are doing so with 2 expansion functions (or modes or PWS dipoles) taken at a time. That is, all the others are open circuited. In this sense, the calculation of $[Z_{mn}]$ is related to Method B in part (a) above.

7.10-1

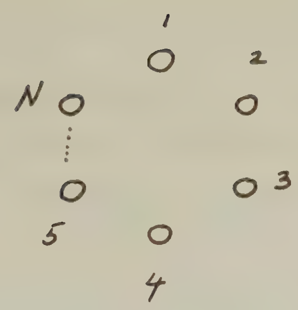
$S_{12} = S_{1N}$

$S_{13} = S_{1(N-1)}$

$S_{14} = S_{1(N-2)}$

$S_{15} = S_{1(N-3)}$

ETC.



$S_{21} = S_{12} = S_{1N}$

$S_{31} = S_{13} = S_{1(N-1)}$

$\vdots$

$S_{N1} = S_{1N} = S_{12}$

$\therefore$ Last row in $[Z]_{array}$ is

$S_{12} \quad S_{13} \quad S_{14} \cdots S_{1N} \quad S_{11}$

and (7-106) follows.

7.11-1

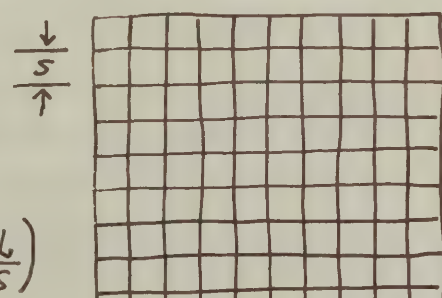
$$\sum_{j=1}^{L\cdot M} Z_{kj} I_j = -E_k^i \,, \quad k = 1, 2, \cdots L\cdot M$$

$$I_j = I_{(j+m)} = I_{(j+2m)} = \cdots = I_{(j+(L-1)M)}$$

$$\sum_{j=1}^{M} I_j Z_{kj} + \sum_{j=M+1}^{2M} I_{(j-m)} Z_{kj} + \cdots = -E_k^i$$

$$\therefore \sum_{j=1}^{M} I_j \left(\sum_{n=0}^{L-1} Z_{k(j+nM)} \right) = -E_k^i \,, \quad k = 1, 2, \cdots M$$

7.11-2



If $s = 0.1\lambda$, no. seg. = 220

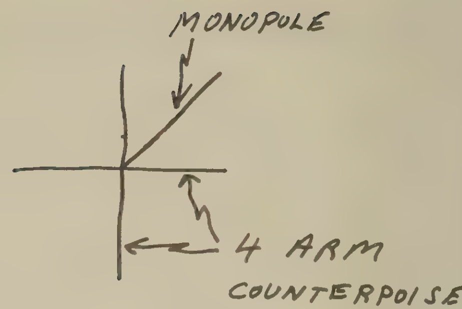
If $s = 0.2\lambda$, no. seg. = 60

In general, no. seg $= 2 \times \left(\frac{L}{s} + 1\right)\left(\frac{L}{s}\right)$

where L = length of one side

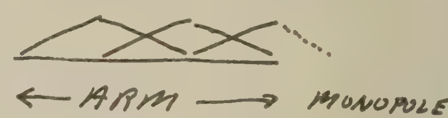
7.11-3

SIMPLEST CONFIGURATION IS TO MODEL CIRCULAR GROUND PLANE WITH 4 ARMS.



a) FOR PULSE FUNCTIONS:
TAKE 20 SEGMENTS PER ARM + AT LEAST 20 ON THE MONOPOLE (SEE PG. 316 AS A GUIDE.) TOTAL = (100+) SEGMENTS

b) FOR PWS FUNCTIONS:
TAKE 3 SEGMENTS PER ARM AND 3 ON THE MONOPOLE.



THUS THE ARMS WILL USE $3+3+3+2=11$ EXPANSION FUNCTIONS DUE TO THE JUNCTION TREATMENT. TOTAL $= 11+3 = 14$

7.11-4

$$(\nabla^2+\beta^2)E_z = j\omega\mu J_z \quad , \quad E_z = E_z(x,y)$$

THE GREEN'S FUNCTION SATIFIES THE 2-D EQ.

$$(\nabla^2+\beta^2)G(\bar{\rho},\bar{\rho}') = -\delta(\bar{\rho}-\bar{\rho}') \quad , \quad \text{SO}$$

$$G(\rho,\rho') = \frac{1}{4j} H_0^{(2)}(\beta|\bar{\rho}-\bar{\rho}'|) \;,\; |\bar{\rho}-\bar{\rho}'| = \sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}$$

WE CAN CONSTRUCT A SOLUTION FOR AN ARBITRARY TWO-DIMENSIONAL DISTRIBUTION OF CURRENT BY DIVIDING THE SOURCE INTO ELEMENTAL FILAMENTS OF CURRENT AND SUMMING THE FIELDS FROM THE ELEMENTS.

$$\therefore \; dA_z = \frac{J_z\,ds'}{4j} H_0^{(2)}(\beta|\bar{\rho}-\bar{\rho}'|)$$

WHERE ds' IS AN AREA PERPENDICULAR TO z.

7.11-4 (CONT.) SUMMING, WE HAVE

$$A_z = \frac{1}{4j}\iint J_z(\bar{\rho}')\,H_0^{(2)}(\beta|\bar{\rho}-\bar{\rho}'|)\,ds'$$

BECAUSE THIS IS A 2-D PROBLEM,

$E_z^s = -j\omega\mu_0 A_z$, THUS

$$E_z^s = \frac{-j\omega\mu_0}{4j}\iint J_z(\bar{\rho}')\,H_0^{(2)}(\beta|\bar{\rho}-\bar{\rho}'|)\,ds'$$

$$E_z^s = \frac{-\beta\eta}{4}\int_C J_z(\bar{\rho}')\,H_0^{(2)}(\beta|\bar{\rho}-\bar{\rho}'|)\,dl'$$

$$E_z^s + E_z^i = 0$$

$$\therefore\ E_z^i(\rho) = \frac{\beta\eta}{4}\int_C J_z(\bar{\rho}')\,H_0^{(2)}(\beta|\bar{\rho}-\bar{\rho}'|)\,dl'$$

7.11-5

$H_z = H_z^i + H_z^s$ = TOTAL MAGNETIC FIELD AT THE SURFACE

$\bar{H} = \nabla\times\bar{A}$ ALLOWS US TO WRITE

$$H_z^s(\rho) = \hat{z}\cdot\nabla\times\int_C J_\phi(\rho)\,\Psi(\rho,\rho')\,dl'$$

H_z IS FINITE EXTERNAL TO C, ZERO INTERNAL TO C, AND THE DISCONTINUITY OF H_z ON C EQUALS THE CURRENT DENSITY. IF THE INTERIOR OF C LIES ON THE LEFT SIDE OF $d\bar{l}$ (RIGHT HAND RULE)

$J_\phi = -[H_z]_{C+}$, C_+ MEANS EVALUATION IS JUST EXTERNAL TO C.

$$\therefore\ J_\phi(\rho) = -\left[H_z^i(\rho) + \hat{z}\cdot\nabla\times\int_C J_\phi(\rho)\,\Psi(\rho,\rho')\,dl'\right]_{C+}$$

7.11-6 FROM (7-125)

$$J_\phi(\rho) + \left[\hat{z}\cdot\nabla\times\int_C J_\phi(\rho) H_0^{(2)}(|\bar{\rho}-\bar{\rho}'|)\,dl' = H_z^i\right|_{c+}$$

USING PULSE FUNCTIONS

$$J_\phi(\rho) = \sum_{n=1}^{N} I_n P(\bar{\rho}-\bar{\rho}_n) \quad \text{WHERE} \quad P(\bar{\rho}-\bar{\rho}_n) = \begin{cases} 1, & |\bar{\rho}-\bar{\rho}_n| < \frac{\Delta C_n}{2} \\ 0, & \text{ELSEWHERE} \end{cases}$$

$$\sum_{n=1}^{N} I_n P(\bar{\rho}-\bar{\rho}_n) + \left[\hat{z}\cdot\nabla\times\int_C \sum_{n=1}^{N} I_n P(\bar{\rho}-\bar{\rho}_n) H_0^{(2)}(\beta|\bar{\rho}-\bar{\rho}'|)\,dl'\right]_{C+} = H_z^i\Big|_{C+}$$

MULTIPLYING BOTH SIDES BY $\delta(\rho-\rho_m)$ AND INTEGRATING OVER C AND DIVIDING BY ΔC_i LEAVES THE FIRST TERM AS

$$\sum_{n=1}^{N} \int_C I_n P(\bar{\rho}-\bar{\rho}_n)\cdot\delta(\bar{\rho}-\bar{\rho}_n)\,dl'/\Delta C_i = \sum_{n=1}^{N} I_n P(\rho_m-\rho_n)$$

AND LEAVES THE SECOND TERM AS

$$\hat{z}\cdot\nabla\times\int_C \sum_{n=1}^{N} I_n(\bar{\rho}-\bar{\rho}_n)\left[\int_C H_0^{(2)}(\beta|\bar{\rho}-\bar{\rho}'|)\,\delta(\rho-\rho_m)\,\frac{dl'}{\Delta C_i}\right]dl'$$

$$= \hat{z}\cdot\nabla\times\int_C \sum_{n=1}^{N} I_n(\bar{\rho}-\bar{\rho}_n) H_0^{(2)}(\beta|\bar{\rho}_m-\bar{\rho}'|)\,dl'$$

$$= \hat{z}\cdot\nabla\times\sum_{n=1}^{N} I_n(\rho_n) H_0^{(2)}(\beta|\bar{\rho}_m-\bar{\rho}_n|) = \hat{z}\sum_{n=1}^{N} I_n H_z(m,n)$$

$H_z(m,n)$ IS THE MAGNETIC FIELD FROM SOURCE I_n

$$\sum_{n=1}^{N} I_n P(\bar{\rho}_m-\bar{\rho}_n) + \hat{z}\cdot\sum_{n=1}^{N} I_n H_z(m,n) = H_z^i\Big|_{c-}$$

$$\sum_{n=1}^{N} I_n\left[P(\bar{\rho}_m-\bar{\rho}_n) + H_z(m,n)\right] = H_z^i\Big|_{c-}$$

THUS, $Z_{mn} = \delta_{mn} + H_z(m,n)$

7. 11-6 (CONT)

$$H_z(m,n) = \left[\hat{z}\cdot\nabla\times\int_{\Delta C_n} H_0^{(2)}\left[\beta\sqrt{(x-x_m)^2+(y-y_m)^2}\,dl'\right]\right]_{C+}$$

FOR $\Delta C_n << \lambda$ AND $|\bar{\rho}-\bar{\rho}'| >> \Delta C_n$

$$H_z(m,n) = \frac{\Delta C_n}{4j}\frac{\partial}{\partial n}\left[H_0^{(2)}(\beta\rho)\right]$$

IF ϕ IS THE ANGLE BETWEEN $\hat{\rho}$ AND $\hat{n}$, THEN

$$H_z(m,n) = \frac{j}{4}\beta\,\Delta C_n\,\cos\phi\; H_1^{(2)}(\beta\rho)$$

TO TRANSLATE THIS RESULT FROM THE LOCAL COORDINATE SYSTEM TO ONE WITH ARBITRARY ORIGIN, WE REPLACE ρ by $|\bar{\rho}_m-\bar{\rho}_n|$ AND $\cos\phi$ by $\hat{n}\cdot\hat{R}$ WHEN

$$\hat{R} = \frac{\bar{\rho}_m-\bar{\rho}_n}{|\bar{\rho}_m-\bar{\rho}_n|}$$

THEN FOR $m\neq n$, WE HAVE

$$Z_{mn} \simeq \frac{j}{4}\beta\,\Delta C_n\,(\hat{n}\cdot\hat{R})\,H_1^{(2)}(\beta|\bar{\rho}_m-\bar{\rho}_n|)$$

---

7. 11-7

a)

b) $$Z_{mn} = \int_{-L/2}^{L/2}\bar{W}_m(l)\cdot\bar{E}_n^s(l)\,dl = E_n^s(l_m),\quad W_m(l)=\delta(l-l_m)$$

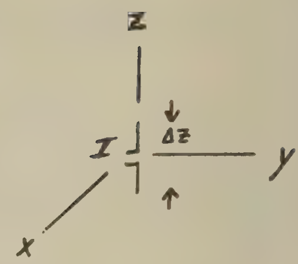
USING THE RESULT IN PROB 7.1-1

$$Z_{mn} = \frac{1}{4\pi j\omega\epsilon_0}\int_{-L/2}^{L/2}\cos(2n-1)\frac{\pi z'}{L}\,\frac{e^{-j\beta R}}{R^5}\left[(1+j\beta R)(2R^2-3a^2)+\beta^2a^2R^2\right]dz'$$

WHERE $R=\sqrt{a^2+(z_m-z')^2}$

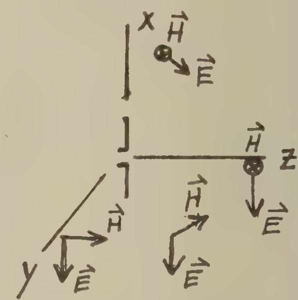
NOTE: INTEGRATION IS FROM $-L/2$ TO $+L/2$

CHAPTER 8



8.1-1 First consider an ideal dipole.
From (1-60) $$\vec{A} = \frac{I e^{-j\beta r}}{4\pi r} \Delta z \ \hat{z}$$
Using (A-3)
$$\vec{A} = \frac{I \Delta z \, e^{-j\beta r}}{4\pi r} (\hat{r}\cos\theta - \hat{\theta}\sin\theta)$$
The far-zone electric field from (1-103) is
$$\vec{E} = -j\omega\mu A_\theta \hat{\theta} - j\omega\mu A_\phi \hat{\phi} = j\omega\mu \frac{I\Delta z}{4\pi r} e^{-j\beta r} \sin\theta \, \hat{\theta}$$

Now, change coordinates so $\vec{I} = I\hat{x}$



Then $$\vec{A} = \frac{I e^{-j\beta r}}{4\pi r} \Delta z \ \hat{x}$$
$$= \frac{I\Delta z \, e^{-j\beta r}}{4\pi r} (\hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi) \quad \text{from (A-1)}$$
So
$$\vec{E} = -j\omega\mu A_\theta \hat{\theta} - j\omega\mu A_\phi \hat{\phi} = j\omega\mu \frac{I\Delta z}{4\pi r} (-\cos\theta\cos\phi\,\hat{\theta} + \sin\phi\,\hat{\phi})$$
which shows that changing coordinates introduces an E_ϕ.

8.1-2 In $$\vec{A} = \frac{e^{-j\beta r}}{4\pi r} \iint_S \vec{J}_s(\vec{r}\,') e^{j\beta \hat{r}\cdot\vec{r}\,'} dS' \qquad (8\text{-}3)$$
replace
$\vec{J}_s$ by $\vec{M}_s$ and $\vec{A}$ by $\vec{F}$ (see Table 2-1).
Then
$$\vec{F} = \frac{e^{-j\beta r}}{4\pi r} \iint_S \vec{M}_s(\vec{r}\,') e^{j\beta \hat{r}\cdot\vec{r}\,'} dS' \quad \text{which is (8-5).}$$
In
$$\vec{E}_A = -j\omega\mu\vec{A} \qquad (8\text{-}4)$$
replace $\vec{A}$ by $\vec{F}$ and $\vec{E}_A$ by $\vec{H}_F$ and μ by ϵ
Then $\vec{H}_F = -j\omega\epsilon\vec{F}$ which is (8-6).

8.1-3 In $\vec{E} = -j\omega\mu\vec{A} - j\omega\epsilon\eta\vec{F}\times\hat{r}$ (8-7)
substituting (8-22) and (8-23) gives

$$\vec{E} = -j\omega\mu\frac{e^{-j\beta r}}{4\pi r}\left[\hat{\theta}\cos\theta(Q_x\sin\phi - Q_y\cos\phi) + \hat{\phi}(Q_x\cos\phi + Q_y\sin\phi)\right]$$

$$-j\omega\epsilon\eta(-1)\frac{e^{-j\beta r}}{4\pi r}\left[\hat{\theta}\cos\phi(P_x\sin\phi - P_y\cos\phi) + \hat{\phi}(P_x\cos\phi + P_y\sin\phi)\right]\times\hat{r}$$

$$= j\frac{e^{-j\beta r}}{4\pi r}\Big\{\hat{\theta}\left[-\omega\mu\cos\theta(Q_x\sin\phi - Q_y\cos\phi) + \omega\epsilon\eta(P_x\cos\phi + P_y\sin\phi)\right]$$

$$-\hat{\phi}\left[\omega\mu(Q_x\cos\phi + Q_y\sin\phi) - \omega\epsilon\eta\cos\theta(P_x\sin\phi - P_y\cos\phi)\right]\Big\}$$

But $\omega\epsilon\eta = \omega\epsilon\sqrt{\frac{\mu}{\epsilon}} = \omega\sqrt{\mu\epsilon} = \beta$ and $\omega\mu = \frac{\beta}{\omega\sqrt{\mu\epsilon}}\omega\mu = \beta\sqrt{\frac{\mu}{\epsilon}} = \beta\eta$

So

$$E_\theta = j\beta\frac{e^{-j\beta r}}{4\pi r}\left[P_x\cos\phi + P_y\sin\phi + \eta\cos\theta(Q_y\cos\phi - Q_x\sin\phi)\right]$$

$$E_\phi = j\beta\frac{e^{-j\beta r}}{4\pi r}\left[\cos\theta(P_y\cos\phi - P_x\cos\phi) - \eta(Q_y\sin\phi + Q_x\cos\phi)\right]$$

which is (8-24).

8.1-4 Eqn. (8-11) is $\vec{H} = -j\omega\epsilon F_\theta\hat{\theta} - j\omega\epsilon F_\phi\hat{\phi}$

The magnetic potential is that of (8-23), but with a factor of 2 arising from (8-10):

$$\vec{F} = -\frac{e^{-j\beta r}}{2\pi r}\left[\hat{\theta}\cos\theta(P_x\sin\phi - P_y\cos\phi) + \hat{\phi}(P_x\cos\phi + P_y\sin\phi)\right]$$

Substituting this into (8-11) gives

$$\vec{H} = -j\omega\epsilon\hat{\theta}\left(-\frac{e^{-j\beta r}}{2\pi r}\right)\left[\cos\theta(P_x\sin\phi - P_y\cos\phi)\right]$$

$$-j\omega\epsilon\hat{\phi}\left(-\frac{e^{-j\beta r}}{2\pi r}\right)(P_x\cos\phi + P_y\sin\phi)$$

$$= j\omega\epsilon\frac{e^{-j\beta r}}{2\pi r}\left[\hat{\theta}\cos\theta(P_x\sin\phi - P_y\cos\phi) + \hat{\phi}(P_x\cos\phi + P_y\sin\phi)\right]$$

Now $\vec{E} = \eta\vec{H}\times\hat{r}$, so

$$\vec{E} = j\omega\epsilon\eta\frac{e^{-j\beta r}}{2\pi r}\left[-\hat{\phi}\cos\theta(P_x\sin\phi - P_y\cos\phi) + \hat{\theta}(P_x\cos\phi + P_y\sin\phi)\right]$$

And $\omega\epsilon\eta = \omega\epsilon\sqrt{\mu/\epsilon} = \omega\sqrt{\mu\epsilon} = \beta$, so

$$E_\theta = j\beta\frac{e^{-j\beta r}}{2\pi r}(P_x\cos\phi + P_y\sin\phi);\ E_\phi = j\beta\frac{e^{-j\beta r}}{2\pi r}\cos\theta(P_y\cos\phi - P_x\sin\phi)$$

which is (8-26).

8.1-5 $\vec{J}_s = \hat{n}\times\vec{H}_a = \hat{z}\times\vec{H}_a = H_{ax}\hat{y} - H_{ay}\hat{x}$ from (8-12)

$$= H_{ax}(\hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi)$$
$$- H_{ay}(\hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi) \quad (1)$$

And

$\vec{A} \sim F.T.(\vec{J}_s)$ from (8-3)

$\vec{E} = -j\omega\mu\vec{A}$ from (8-4)

So $\vec{E} \sim -\vec{A} \sim F.T.(\vec{J}_s)$

The contribution of $\vec{J}_s$ to the far-zone field $\vec{E}$ are the θ,ϕ-components of (1). At the same time noting that $\vec{Q} = F.T.(\vec{H}_a)$,

$$\vec{E} \sim \cos\theta(Q_y\cos\phi - Q_x\sin\phi)\hat{\theta} - (Q_y\sin\phi + Q_x\cos\phi) \quad (2)$$

Similarly

$$\vec{M}_s = \vec{E}_a\times\hat{n} = \vec{E}_a\times\hat{z} = -E_{ax}\hat{y} + E_{ay}\hat{x} \quad \text{from (8-13)}$$
$$= -E_{ax}(\hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi)$$
$$+ E_{ay}(\hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi) \quad (3)$$

And

$\vec{F} \sim F.T.(\vec{M}_s)$ from (8-5)

$\vec{H} = -j\omega\epsilon\vec{F}$ from (8-6)

So $\vec{H} \sim -\vec{F} \sim F.T.(\vec{M}_s)$

Noting $\vec{P} = F.T.(\vec{H}_a)$ and retaining θ,ϕ-components, (3) leads to

$$\vec{H} \sim \cos\theta(P_x\sin\phi - P_y\cos\phi)\hat{\theta} + (P_x\cos\phi + P_y\sin\phi)\hat{\phi} \quad (4)$$

The corresponding electric field is

$$\vec{E} = \eta\vec{H}\times\hat{r} \sim \vec{H}\times\hat{r}$$

This with (4) gives

$$\vec{E} \sim (P_x\cos\phi + P_y\sin\phi)\hat{\theta} + \cos\theta(P_y\cos\phi - P_x\sin\phi)\hat{\phi} \quad (5)$$

Now, (2) and (5) account for the trigonometric functions in (8-24)

8.1-6 $\vec{E}_a = \hat{x} E_o \quad |y| \le L/2$

Similar to Example 8-1

$$\vec{P} = \hat{x} E_o L \frac{\sin(\frac{\beta L}{2}\sin\theta)}{\frac{\beta L}{2}\sin\theta} \qquad \text{in } yz\text{-plane } (\phi = 90°)$$

From (8-26) with $\phi = 90°$

$$E_\theta = 0 \qquad E_\phi = -j\beta \frac{e^{-j\beta r}}{2\pi r} \cos\theta \frac{\sin(\frac{\beta L}{2}\sin\theta)}{\frac{\beta L}{2}\sin\theta} E_o L$$

Then

$$F(\theta) = \cos\theta \frac{\sin(\frac{\beta L}{2}\sin\theta)}{\frac{\beta L}{2}\sin\theta}$$

which compares to (4-17) with a coordinate change

E_ϕ at $\theta = 90°$ is tangent to the conducting plane and must be zero. It is because $\cos 90° = 0$.

8.2-1

$$P_y = E_o \int_{-Lx/2}^{Lx/2} e^{j\beta \sin\theta\cos\phi\, x'} dx' \int_{-Ly/2}^{Ly/2} e^{j\beta\sin\theta\sin\phi\, y'} dy'$$

$$= E_o \frac{e^{j\frac{\beta L_x}{2}\sin\theta\cos\phi} - e^{-j\frac{\beta L_x}{2}\sin\theta\cos\phi}}{j\beta\sin\theta\cos\phi} \cdot \frac{e^{j\frac{\beta L_y}{2}\sin\theta\sin\phi} - e^{-j\frac{\beta L_y}{2}\sin\theta\sin\phi}}{j\beta\sin\theta\sin\phi}$$

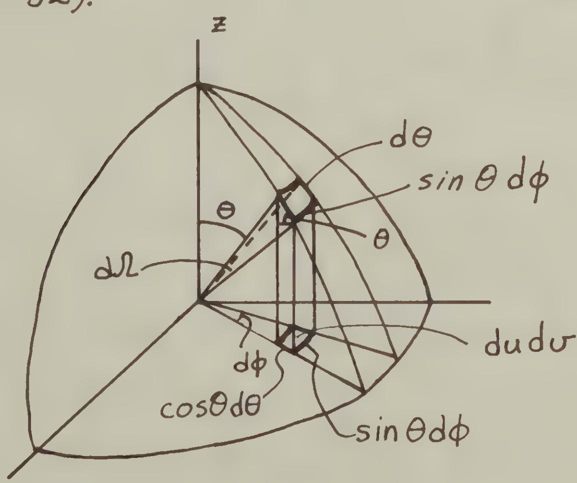
$$= E_o L_x L_y \frac{\sin(\frac{\beta L_x}{2}\sin\theta\cos\phi)}{\frac{\beta L_x}{2}\sin\theta\cos\phi} \frac{\sin(\frac{\beta L_y}{2}\sin\theta\sin\phi)}{\frac{\beta L_y}{2}\sin\theta\sin\phi}$$

which is (8-32).

8.2-2

$$dS = r^2 d\Omega$$
$$= d\Omega \text{ for } r = 1$$
$$= \sin\theta\, d\theta\, d\phi$$

$$du\,dv = \cos\theta\sin\theta\, d\theta\, d\phi$$
$$= \cos\theta\, d\Omega$$



8.3-1 From (8-59) $U(\theta,\phi) = \frac{1}{2\eta}\left[|E_\theta|^2 + |E_\phi|^2\right] r^2$

And

$$E_\theta = j\beta \frac{e^{-j\beta r}}{4\pi r}\left[P_x\cos\phi + P_y\sin\phi + \eta\cos\theta(Q_y\cos\phi - Q_x\sin\phi)\right]$$
$$E_\phi = j\beta \frac{e^{-j\beta r}}{4\pi r}\left[\cos\theta(P_y\cos\phi - P_x\sin\phi) - \eta(Q_y\sin\phi + Q_x\cos\phi)\right] \quad (8\text{-}24)$$

With

$$Q_x = -P_y/\eta \text{ and } Q_y = P_x/\eta \qquad (8\text{-}61)$$

give

$$E_\theta = j\beta \frac{e^{-j\beta r}}{4\pi r}\left[P_x\cos\phi + P_y\sin\phi + \cos\theta(P_x\cos\phi + P_y\sin\phi)\right]$$
$$E_\phi = j\beta \frac{e^{-j\beta r}}{4\pi r}\left[\cos(P_y\cos\phi - P_x\sin\phi) - (P_x\sin\phi - P_y\cos\phi)\right]$$

So

$$U(\theta,\phi) = \frac{1}{2\eta}\frac{\beta^2}{(4\pi r)^2}(1+\cos\theta)^2\left[|P_x\cos\phi + P_y\sin\phi|^2 + |P_y\cos\phi - P_x\sin\phi|^2\right] r^2$$
$$= \frac{\beta^2}{2\eta\cdot 16\pi^2}(1+\cos\theta)^2\left[(P_x\cos\phi + P_y\sin\phi)(P_x^*\cos\phi + P_y^*\sin\phi)\right.$$
$$\left. + (P_y\cos\phi - P_x\sin\phi)(P_y^*\cos\phi - P_x^*\sin\phi)\right]$$
$$= \frac{\beta^2}{32\pi^2\eta}(1+\cos\theta)^2\left[|P_x|^2\cos^2\phi + |P_y|^2\sin^2\phi + (P_x P_y^* + P_y P_x^*)\sin\phi\cos\phi\right.$$
$$\left. + |P_y|^2\cos^2\phi + |P_x|^2\sin^2\phi + (-P_y P_x^* - P_x P_y^*)\sin\phi\cos\phi\right]$$
$$= \frac{\beta^2}{32\pi^2\eta}(1+\cos\theta)^2\left[|P_x|^2 + |P_y|^2\right] \quad \text{which is (8-62).}$$

8.3-2 Let $f = \vec{E}_a$ and $g = 1$

Then

$$\left|\iint_S fg\, dS'\right|^2 \le \iint_S f^2 dS' \iint_S g^2 dS' \quad \text{(Schwartz inequality)}$$

becomes

$$\left|\iint_{S_a} \vec{E}_a\, dS'\right|^2 \le \iint_{S_a} |\vec{E}_a|^2 dS' \iint_{S_a} 1^2 dS' = \iint_{S_a} |\vec{E}_a|^2 dS'\, A_p$$

or

$$\frac{\left|\iint_{S_a} \vec{E}_a\, dS'\right|^2}{\iint_{S_a} |\vec{E}_a|^2 dS'} \le A_p \; ; \; D = \frac{4\pi}{\lambda^2}\frac{\left|\iint_{S_a} \vec{E}_a\, dS'\right|^2}{\iint_{S_a} |\vec{E}_a|^2 dS'} \le \frac{4\pi}{\lambda^2} A_p = D_u$$

D is maximum for a uniform excitation, since $\vec{E}_a = E_o\hat{u}$ gives

$$\left|\iint_{S_a} \vec{E}_a\, dS'\right|^2 = |\hat{u}\, E_o A_p|^2 = E_o^2 A_p^2 \text{ and } \iint_{S_a} |\vec{E}_a|^2 dS' = E_o^2 A_p$$

So $D_u = \frac{4\pi}{\lambda^2}\frac{E_o^2 A_p^2}{E_o^2 A_p} = \frac{4\pi}{\lambda^2} A_p \quad \therefore \boxed{D \le D_u}$

8.3-3 Let $\vec{E}_a = \hat{y} E_o \cos^2 \frac{\pi x'}{a}$ for $-\frac{a}{2} \le x' \le \frac{a}{2}$, $-\frac{b}{2} \le y' \le \frac{b}{2}$

The directivity follows from

$$D = \frac{4\pi}{\lambda^2} \frac{\left|\iint_{S_a} \vec{E}_a \, dS'\right|^2}{\iint_{S_a} |\vec{E}_a|^2 dS'} \qquad (8\text{-}65)$$

First $\iint_{S_a} |\vec{E}_a|^2 dS' = E_o^2 \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \cos^4 \frac{\pi x'}{a} dx' dy'$

$= E_o^2 (b) \int_{-\pi/2}^{\pi/2} \cos^4 t \, dt \left(\frac{a}{\pi}\right)$ where $t = \frac{\pi x'}{a}$ and $dt = \frac{\pi}{a} dx'$

$= E_o^2 \, b \frac{a}{\pi} 2 \int_0^{\pi/2} \left[\frac{\cos 4t}{8} + \frac{\cos 2t}{2} + \frac{3}{8}\right] dt$ where (B-14) and (B-11) were used

$= \frac{2ab}{\pi} E_o^2 \left[\frac{\sin 4t}{32} + \frac{\sin 2t}{4} + \frac{3}{8} t\right]_0^{\pi/2}$

$= \frac{2ab}{\pi} E_o^2 \left(\frac{3}{8} \frac{\pi}{2}\right) = \frac{3}{8} ab E_o^2$

And $\left|\iint_{S_a} \vec{E}_a \, dS'\right| = \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} E_o \cos^2 \frac{\pi x'}{a} dx' dy' = E_o (b) \int_{-\pi/2}^{\pi/2} \cos^2 t \, dt \frac{a}{\pi}$

$= E_o \frac{ab}{\pi} 2 \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2t) dt = E_o \frac{ab}{\pi} \left[t + \frac{\sin 2t}{2}\right]_0^{\pi/2}$

$= \frac{ab}{\pi}\left(\frac{\pi}{2}\right) E_o = \frac{ab}{2} E_o$

Thus

$$D = \frac{4\pi}{\lambda^2} \frac{\left(\frac{ab}{2} E_o\right)^2}{\frac{3}{8} ab E_o^2} = \frac{4\pi}{\lambda^2} \frac{2}{3} ab$$

So

$$\epsilon_t = D/D_u = \frac{4\pi}{\lambda^2} \frac{2}{3} ab \Big/ \frac{4\pi}{\lambda^2} ab = \boxed{\frac{2}{3}}$$

8.3-4 Let $\vec{E}_a = \hat{u} E_o \cos \frac{\pi x'}{L_x} \cos \frac{\pi y'}{L_y}$ $\quad -\frac{L_x}{2} \le x' \le \frac{L_x}{2}$, $-\frac{L_y}{2} \le y' \le \frac{L_y}{2}$

First

$\left|\iint_{S_a} \vec{E}_a \, dS'\right| = E_o \int_{-L_x/2}^{L_x/2} \cos \frac{\pi x'}{L_x} dx' \int_{-L_y/2}^{L_y/2} \cos \frac{\pi y'}{L_y} dy'$

$= E_o \left[\frac{\sin \frac{\pi x'}{L_x}}{\pi/L_x}\right]_{-L_x/2}^{L_x/2} \left[\frac{\sin \frac{\pi y'}{L_y}}{\pi/L_y}\right]_{-L_y/2}^{L_y/2} = E_o \left(\frac{L_x}{\pi} 2\right)\left(\frac{L_y}{\pi} 2\right) = \frac{4}{\pi^2} L_x L_y E_o$

And

$\iint_{S_a} |\vec{E}_a|^2 dS' = E_o^2 \int_{-L_x/2}^{L_x/2} \cos^2 \frac{\pi x'}{L_x} dx' \int_{-L_y/2}^{L_y/2} \cos^2 \frac{\pi y'}{L_y} dy'$

$= E_o^2 \left[\frac{x'}{2}\right]_{-L_x/2}^{L_x/2} \left[\frac{y'}{2}\right]_{-L_y/2}^{L_y/2} = E_o^2 \frac{L_x}{2} \frac{L_y}{2}$

8.3-4 (con't) From (8-65)

$$D=\frac{4\pi}{\lambda^2}\frac{\left|\iint_{S_a}\vec{E}_a\,dS'\right|^2}{\iint_{S_a}|\vec{E}_a|^2 dS'}=\frac{4\pi}{\lambda^2}\frac{\left(\frac{4}{\pi^2}L_xL_y\right)^2E_o^2}{\frac{L_x}{2}\frac{L_y}{2}E_o^2}=\frac{256}{\pi^3}\frac{L_xL_y}{\lambda^2}$$

or $D=\frac{4\pi}{\lambda^2}\frac{64}{\pi^4}L_xL_y=\frac{4\pi}{\lambda^2}(0.657)L_xL_y=\frac{4\pi}{\lambda^2}(0.657)A_p$

But
$D=\epsilon_t D_u=\epsilon_t\frac{4\pi}{\lambda^2}A_p$ So $\boxed{\epsilon_t=0.657}$

8.3-5 a) $D_u=\frac{4\pi}{\lambda^2}A_p=\frac{4\pi}{\lambda^2}L_xL_y=\frac{4\pi}{\lambda^2}(10\lambda)(20\lambda)=4\pi(200)=2513$

$D_u(\text{dB})=10\log D_u=10\log 2513=\boxed{34.0\text{ dB}}$

b) $D=\epsilon_t D_u=\frac{8}{\pi^2}(2513)=2037.18=\boxed{33.1\text{ dB}}$

8.3-6

$$\left|\iint_{S_a}\vec{E}_a\,dS'\right|^2=\left\{\int_{-L_x/2}^{L_x/2}\left(1-\frac{2}{L_x}|x'|\right)dx'\int_{-L_y/2}^{L_y/2}dy'\right\}^2=L_y^2\left[2\int_0^{L_x/2}\left(1-\frac{2}{L_x}x'\right)dx'\right]^2$$

$$=L_y^2\,4\left[\left(x'-\frac{2}{L_x}\frac{x'^2}{2}\right)_{x'=0}^{x'=L_x/2}\right]^2=L_y^2\,4\left(\frac{L_x}{4}\right)^2=L_x^2L_y^2/4$$

$$\iint_{S_a}|\vec{E}_a|^2dS'=\int_{-L_x/2}^{L_x/2}\left[1-\frac{2}{L_x}|x'|\right]^2dx'\int_{-L_y/2}^{L_y/2}1\,dy'=2\int_0^{L_x/2}\left(1-2\frac{2}{L_x}x'+\frac{4}{L_x^2}x'^2\right)dx'\,L_y$$

$$=2L_y\left[x'-\frac{4}{L_x}\frac{x'^2}{2}+\frac{4}{L_x^2}\frac{x'^3}{3}\right]_0^{L_x/2}=2L_y\left(\frac{L_x}{6}\right)=L_xL_y/3$$

So, from (8-65),

$$D=\frac{4\pi}{\lambda^2}\frac{\left|\iint_{S_a}\vec{E}_a\,dS'\right|^2}{\iint_{S_a}|\vec{E}_a|^2dS'}=\frac{4\pi}{\lambda^2}\frac{L_x^2L_y^2/4}{L_xL_y/3}=\frac{4\pi}{\lambda^2}\left(\frac{3}{4}L_xL_y\right)=\frac{4\pi}{\lambda^2}\left(\frac{3}{4}A_p\right)$$

$=\frac{3}{4}D_u$ and $D=\epsilon_t D_u$ $\therefore$ $\boxed{\epsilon_t=3/4}$

8.3-7

Strictly speaking, no. This because the aperture electric field must have a component somewhere which is tangent to the aperture opening. This aperture edge is a perfect conductor which requires a zero tangential field there, thus violating the uniform excitation condition. In practice, however, for apertures large relative to a wavelength the edge effects are minor.

8.3-8 $f = 150\,MHz \Rightarrow \lambda = 2m$; $G = 23dB \Rightarrow G = 199.526$; $D = 23.5\,dB \Rightarrow D = 223.872$

(a) $G = \frac{4\pi}{\lambda^2} A_e \Rightarrow A_e = \frac{\lambda^2}{4\pi} G = \frac{(2)^2}{4\pi}(199.526) = \boxed{63.51\ m^2}$

(b) $D = \frac{4\pi}{\lambda^2} A_{em} \Rightarrow A_{em} = \frac{\lambda^2}{4\pi} D = \frac{(2)^2}{4\pi}(223.872) = \boxed{71.26\ m^2}$

(c) $G = \frac{4\pi}{\lambda^2} \epsilon_{ap} A_p \Rightarrow \epsilon_{ap} = \left(\frac{\lambda^2}{4\pi} G\right) / A_p = A_e / A_p = 63.51/100 = \boxed{0.6351}$

(d) $G = eD \Rightarrow e = G/D = 199.526/223.872 = \boxed{0.89125}$

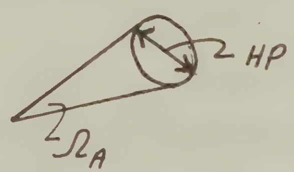
(e) $\frac{D}{D_u} = \frac{D}{\frac{4\pi}{\lambda^2} A_p} = \frac{223.872}{\frac{4\pi}{(2)^2} 100} = \boxed{0.7126}$

Note $(e) \cdot \left(\frac{D}{D_u}\right) = (0.89125) \cdot (0.7126) = 0.6351 = \epsilon_{ap}$ ✓

8.3-9

$$D \approx \frac{4\pi}{HP_E\, HP_H} = \frac{4\pi}{\frac{\pi}{180} HP_{E^\circ} \frac{\pi}{180} HP_{H^\circ}} = \frac{4\pi\left(\frac{180}{\pi}\right)^2}{HP_{E^\circ}\ HP_{H^\circ}} = \frac{41{,}253}{HP_{E^\circ}\ HP_{H^\circ}}$$

8.3-10 When $HP_E = HP_H = HP$ we can approximate the beam solid angle as



Then

$$\Omega_A = \int_0^{2\pi}\int_0^{HP/2} \sin\theta\, d\theta\, d\phi = \int_0^{2\pi} d\phi \int_0^{HP/2} \sin\theta\, d\theta = 2\pi\left[-\cos\theta\right]_0^{HP/2}$$

$$= 2\pi\left(-\cos\frac{HP}{2} + 1\right) = 2\pi\left(2\sin^2\frac{HP}{4}\right) \approx 4\pi\left(\frac{HP}{4}\right)^2 = \frac{\pi}{4} HP^2 \quad \text{for HP small}$$

And

$$D = \frac{4\pi}{\Omega_A} \approx \frac{4\pi}{\frac{\pi}{4} HP^2} = \frac{16}{HP^2} = \frac{16}{\left(\frac{\pi}{180} HP_{deg}\right)^2} = \boxed{\frac{52{,}525}{(HP_{deg})^2}}$$

which is (8-81) with $HP_{E^\circ} = HP_{H^\circ} = HP_{deg} = HP$ in degrees.

8.3-11 $f = 0.44\,GHz$, $\lambda = 3\times10^{10}/0.44\times10^9 = 68.18\,cm$, $HP_{E^\circ} = 30^\circ$, $HP_{H^\circ} = 27^\circ$
$G = 15.5\,dB$, $G = 10^{1.55} = 35.48$, $A_p = (185.5)(137.4) = 25{,}487.7\ cm^2$

(a) From (8-79)

$$\epsilon_{ap} = \frac{\lambda^2}{4\pi}\frac{G}{A_p} = \frac{(68.18)^2}{4\pi}\frac{35.48}{25{,}487.7} = \boxed{0.515}$$

8.3-11 (con't)

(b) From (8-82) $G \approx \dfrac{26{,}000}{HP_{E^\circ} HP_{H^\circ}} = \dfrac{26{,}000}{(30)(27)} = 32.1 = \boxed{15.1\,\text{dB}}$

(SA model 12-0.4 horn)

8.3-12 $f = 6.3$ GHz, $\lambda = 30/6.3 = 4.76$ cm, $HP_{E^\circ} = 12°$, $HP_{H^\circ} = 13°$,

$G = 22.1$ dB, $G = 162.18$, $A_p = (28.85)(21.39) = 617.1\ \text{cm}^2$

(a) $\epsilon_{ap} = \dfrac{\lambda^2}{4\pi}\dfrac{G}{A_p} = \dfrac{(4.76)^2}{4\pi}\dfrac{162.18}{617.1} = \boxed{0.474}$

(b) $G \approx \dfrac{26{,}000}{HP_{E^\circ} HP_{H^\circ}} = \dfrac{26{,}000}{(12)(13)} = 166.67 = \boxed{22.2\,\text{dB}}$

(SA model 12-5.8 horn)

8.3-13 $d = 3.66$ m, $f = 460$ MHz, $\lambda = 300/460 = 0.6522$ m

measured: $G = 22.2$ dB $\quad HP_{E^\circ} = HP_{H^\circ} = 12.5°$

From (8-84) $G = 5.43\dfrac{d^2}{\lambda^2} = 5.43\left(\dfrac{3.66}{0.6522}\right)^2 = 171.015 = \boxed{22.3\,\text{dB}}$

From (8-82) $G = \dfrac{26{,}000}{HP_{E^\circ} HP_{H^\circ}} = \dfrac{26{,}000}{(12.5)^2} = 166.4 = \boxed{22.2\,\text{dB}}$

(Anixter-Mark model P-4144GR)

8.3-14 $f = 28.56$ GHz, $\lambda = 30/28.56 = 1.05$ cm (COMSTAR satellite)

(a) $d = 1.22$ m : $G = 5.43\dfrac{d^2}{\lambda^2} = 5.43\left(\dfrac{122}{1.05}\right)^2 = 73{,}247 = \boxed{48.65\,\text{dB}}$

(b) $HP_{E^\circ} = 0.605°$, $HP_{H^\circ} = 0.556°$: $G = \dfrac{26{,}000}{HP_{E^\circ} HP_{H^\circ}} = \dfrac{26{,}000}{(0.605)(0.556)} = 77{,}293.5 = \boxed{48.88\,\text{dB}}$

(Measured gain = 48.5 dB)

8.4-1

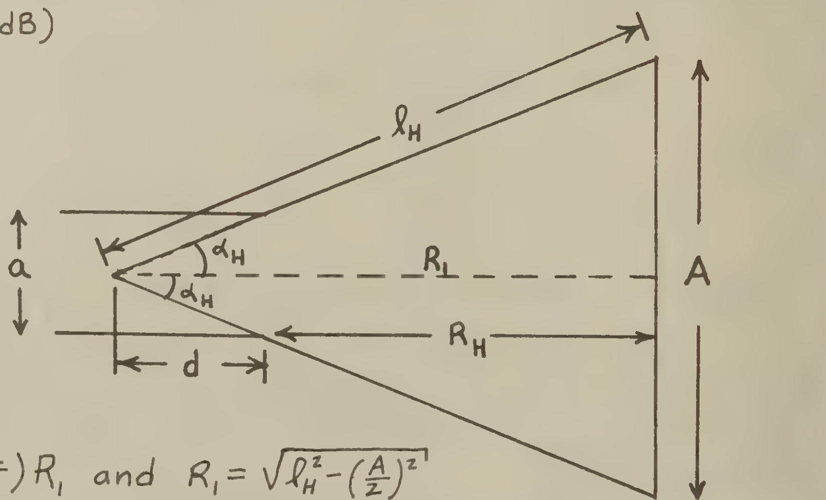
First

$R_H = R_1 - d$

But $\dfrac{a}{2d} = \dfrac{A}{2R_1}$

by similar triangles. Or

$d = \dfrac{aR_1}{A}$



So $R_H = R_1 - \dfrac{aR_1}{A} = \left(1 - \dfrac{a}{A}\right)R_1$ and $R_1 = \sqrt{\ell_H^2 - \left(\dfrac{A}{2}\right)^2}$

Hence

$R_H = \left(1-\dfrac{a}{A}\right)\sqrt{\ell_H^2 + \dfrac{A^2}{4}} = \left(1-\dfrac{a}{A}\right)A\sqrt{\left(\dfrac{\ell_H}{A}\right)^2 - \dfrac{1}{4}} = (A-a)\sqrt{\left(\dfrac{\ell_H}{A}\right)^2 - \dfrac{1}{4}}$ which is (8-86)

8.4-2 The H-plane contribution of (8-92) is

$$\mathcal{I}=\int_{-A/2}^{A/2}\cos\frac{\pi x'}{A}e^{-j\frac{\beta}{2R_1}(x')^2}e^{j\beta ux'}dx'$$

$$=\frac{1}{2}\int_{-A/2}^{A/2}\left(e^{j\frac{\pi x'}{A}}+e^{-j\frac{\pi x'}{A}}\right)e^{-j\frac{\beta}{2R_1}(x')^2}e^{j\beta ux'}dx'$$

$$=\frac{1}{2}\int_{-A/2}^{A/2}e^{j\left[-\frac{\beta}{2R_1}(x')^2+\left(\frac{\pi}{A}+\beta u\right)x'\right]}dx'+\frac{1}{2}\int_{-A/2}^{A/2}e^{j\left[-\frac{\beta}{2R_1}(x')^2+\left(-\frac{\pi}{A}+\beta u\right)x'\right]}dx'$$

Completing the squares in the exponents,

$$\mathcal{I}=\frac{1}{2}\Bigg\{e^{j\frac{R_1}{2\beta}\left(\frac{\pi}{A}+\beta u\right)^2}\int_{-\frac{A}{2}}^{A/2}e^{-j\frac{\beta}{2R_1}\left[x'-\frac{R_1}{\beta}\left(\frac{\pi}{A}+\beta u\right)\right]^2}dx'$$

$$+e^{j\frac{R_1}{2\beta}\left(-\frac{\pi}{A}+\beta u\right)^2}\int_{-\frac{A}{2}}^{\frac{A}{2}}e^{-j\frac{\beta}{2R_1}\left[x'-\frac{R_1}{\beta}\left(-\frac{\pi}{A}+\beta u\right)\right]^2}dx'\Bigg\}$$

Let

$$\frac{\pi}{2}s'^2=\frac{\beta}{2R_1}\left[x'-\frac{R_1}{\beta}\left(\beta u+\frac{\pi}{A}\right)\right]^2 \quad\text{or}\quad s'=\sqrt{\frac{2}{\pi}\frac{\beta}{2R_1}}\left[x'-\frac{R_1}{\beta}\left(\beta u+\frac{\pi}{A}\right)\right]$$

$$\frac{\pi}{2}t'^2=\frac{\beta}{2R_1}\left[x'-\frac{R_1}{\beta}\left(\beta u-\frac{\pi}{A}\right)\right]^2 \quad\text{or}\quad t'=\sqrt{\frac{2}{\pi}\frac{\beta}{2R_1}}\left[x'-\frac{R_1}{\beta}\left(\beta u-\frac{\pi}{A}\right)\right]$$

Then let [which is (8-95)]

$$s_1'=\sqrt{\frac{1}{\pi\beta R_1}}\left[-\frac{\beta A}{2}-R_1\beta u-\frac{\pi R_1}{A}\right];\ s_2'=\sqrt{\frac{1}{\pi\beta R_1}}\left[\frac{\beta A}{2}-R_1\beta u-\frac{\pi R_1}{A}\right]$$

$$t_1'=\sqrt{\frac{1}{\pi\beta R_1}}\left[-\frac{\beta A}{2}-R_1\beta u+\frac{\pi R_1}{A}\right];\ t_2'=\sqrt{\frac{1}{\pi\beta R_1}}\left[\frac{\beta A}{2}-R_1\beta u+\frac{\pi R_1}{A}\right]$$

And using $dx'=\sqrt{\frac{\pi R_1}{\beta}}\,ds'$,

$$\mathcal{I}=\frac{1}{2}\Bigg\{e^{j\frac{R_1}{2\beta}\left(\beta u+\frac{\pi}{A}\right)^2}\int_{s_1'}^{s_2'}e^{-j\frac{\pi}{2}s'^2}\sqrt{\frac{\pi R_1}{\beta}}\,ds'$$

$$+e^{j\frac{R_1}{2\beta}\left(\beta u-\frac{\pi}{A}\right)^2}\int_{t_1'}^{t_2'}e^{-j\frac{\pi}{2}t'^2}\sqrt{\frac{\pi R_1}{\beta}}\,ds'\Bigg\}$$

$$=\frac{1}{2}\sqrt{\frac{\pi R_1}{\beta}}\Bigg\{e^{j\frac{R_1}{2\beta}\left(\beta u+\frac{\pi}{A}\right)^2}\left[\int_{s_1'}^{s_2'}\left(\cos\left(\tfrac{\pi}{2}s'^2\right)-j\sin\left(\tfrac{\pi}{2}s'^2\right)\right)ds'\right]$$

$$+e^{j\frac{R_1}{2\beta}\left(\beta u-\frac{\pi}{A}\right)^2}\int_{t_1'}^{t_2'}\left(\cos\left(\tfrac{\pi}{2}t'^2\right)-j\sin\left(\tfrac{\pi}{2}t'^2\right)\right)dt'\Bigg]\Bigg\}$$

$$=\frac{1}{2}\sqrt{\frac{\pi R_1}{\beta}}\Bigg\{e^{j\frac{R_1}{2\beta}\left(\beta u+\frac{\pi}{A}\right)^2}\left[C(s_2')-jS(s_2')-C(s_1')+jS(s_1')\right]$$

$$+e^{j\frac{R_1}{2\beta}\left(\beta u-\frac{\pi}{A}\right)^2}\left[C(t_2')-jS(t_2')-C(t_1')+jS(t_1')\right]\Bigg\}$$

and

$$\vartheta = \frac{1}{2}\sqrt{\frac{\pi R_1}{\beta}}\, I \quad \text{in (8-93) and (8-94).}$$

8.4-3

a) From (8-95) $\quad s_1' = \sqrt{\frac{1}{\pi\beta R_1}}\left[-\frac{\beta A}{2} - R_1\beta u - \frac{\pi R_1}{A}\right]$

$$s_1' = \sqrt{\frac{A^2}{8\lambda R_1}}\,\frac{1}{A}\,\frac{2}{\pi}\,\lambda\left[-\frac{\beta A}{2} - R_1\beta u - \frac{\pi R_1}{A}\right] = 2\sqrt{t}\left[-1 - \frac{2R_1}{A}u - \frac{\lambda R_1}{A^2}\right]$$

where $t = A^2/8\lambda R_1$ as in (8-102)

$$s_1' = 2\sqrt{t}\left[-1 - \frac{1}{4}\frac{1}{t}\left(\frac{A}{\lambda}\sin\theta\cos\phi\right) - \frac{1}{8}\frac{1}{t}\right]$$

For $\phi = 0°$

$$s_1'(\phi = 0°) = 2\sqrt{t}\left[-1 - \frac{1}{4}\frac{1}{t}\left(\frac{A}{\lambda}\sin\theta\right) - \frac{1}{8t}\right] = s_1 \quad \text{as in (8-104).}$$

b) The phase term in (8-94) is

$$\frac{R_1}{2\beta}\left(\beta\sin\theta\cos\phi + \frac{\pi}{A}\right)^2 = \frac{1}{2}\,\frac{\lambda R_1}{2\pi}\,\frac{1}{A^2}\left(2\pi\frac{A}{\lambda}\sin\theta\cos\phi + \pi\right)^2$$

$$= \pi\,\frac{\lambda R_1}{A^2}\left(\frac{A}{\lambda}\sin\theta\cos\phi + \frac{1}{2}\right)^2 = \frac{\pi}{8}\frac{1}{t}\left(\frac{A}{\lambda}\sin\theta\cos\phi + \frac{1}{2}\right)^2$$

and this for $\phi = 0°$ is

$$\frac{\pi}{8}\frac{1}{t}\left(\frac{A}{\lambda}\sin\theta + \frac{1}{2}\right)^2 \quad \text{which is the phase term in (8-103).}$$

8.4-4

The integral $\left|\iint_{S_a}\vec{E}_a\,dS'\right|^2$ can be found from the Fourier transform of $\vec{E}_a$ with $\theta = 0$ and $\phi = 0$ in (8-93):

$$\left|\iint_{S_a}\vec{E}_a\,dS'\right|^2 = \frac{E_0^2}{4}\,\frac{\pi R_1 b^2}{2\pi/\lambda}\,\left|I(\theta=0,\phi=0)\right|^2$$

And

$$\iint_{S_a}|\vec{E}_a|^2 dS' = E_0^2\int_{-\frac{A}{2}}^{A/2}\cos^2\frac{\pi x'}{A}\,dx'\int_{-\frac{b}{2}}^{\frac{b}{2}}dy' \quad \text{using } \vec{E}_a \text{ in (8-91)}$$

$$= E_0^2 b\int_{-\frac{A}{2}}^{\frac{A}{2}}\frac{1}{2}\left(1 + \cos 2\frac{\pi x'}{A}\right)dx' = E_0^2\frac{b}{2}\left[x' + \frac{\sin\frac{2\pi x'}{A}}{2\pi/A}\right]_{-\frac{A}{2}}^{\frac{A}{2}} = E_0^2\frac{Ab}{2}$$

Now, with $u=0$ in (8-95)

$$s_1' \to -p_1 = \sqrt{\frac{1}{\pi\beta R_1}}\left[-\frac{\beta A}{2} - \frac{\pi R_1}{A}\right];\ s_2' \to -p_2 = \sqrt{\frac{1}{\pi\beta R_1}}\left[\frac{\beta A}{2} - \frac{\pi R_1}{A}\right];\ t_1' \to p_2;\ t_2' \to p_1$$

Then from (8-94)

$$I(\theta=0,\phi=0) = e^{j\frac{R_1}{2\beta}\left(\frac{\pi}{A}\right)^2}\Big[C(-p_2) - jS(-p_2) - C(-p_1) + jS(-p_1)$$
$$+ C(p_1) - jS(p_1) - C(p_2) + jS(p_2)\Big]$$

8.4-4 (con't)

$$|I(\theta=0,\phi=0)|^2 = |C(-p_2) - jS(-p_2) - C(-p_1) + jS(-p_1) + C(p_1) - jS(p_1) - C(p_2) + jS(p_2)|^2$$

$$= 4\left|-C(p_2) + C(p_1) + j[S(p_2) - S(p_1)]\right|^2 \text{ since } C(-x) = -C(x) \text{ \& } S(-x) = -S(x)$$

So

$$D_H = \frac{4\pi}{\lambda^2} \frac{\left|\iint_{S_a} \vec{E}_a \, dS'\right|^2}{\iint_{S_a} |\vec{E}_a|^2 dS'} = \frac{4\pi}{\lambda^2} \frac{\frac{E_o^2}{4} \frac{\pi R_1 b^2}{2\pi/\lambda} 4\left\{[C(p_1) - C(p_2)]^2 + [S(p_1) - S(p_2)]^2\right\}}{E_o^2 \frac{Ab}{2}}$$

$$= \frac{4\pi R_1 b}{\lambda A} \left\{[C(p_1) - C(p_2)]^2 + [S(p_1) - S(p_2)]^2\right\} \quad (8\text{-}105a)$$

and from above

$$p_1 = \frac{1}{\sqrt{\pi \beta R_1}} \left[\frac{\beta A}{2} + \frac{\pi R_1}{A}\right] = \frac{1}{\sqrt{2}} \frac{1}{\pi} \left[\frac{\pi R_1 \sqrt{\lambda}}{A\sqrt{R_1}} + \frac{\pi A \sqrt{\lambda}}{\lambda \sqrt{R_1}}\right] = \frac{1}{\sqrt{2}} \left[\frac{\sqrt{R_1/\lambda}}{A/\lambda} + \frac{A/\lambda}{\sqrt{R_1/\lambda}}\right]$$

$$p_2 = \frac{-1}{\sqrt{\pi \beta R_1}} \left[\frac{\beta A}{2} - \frac{\pi R_1}{A}\right] = \frac{1}{\sqrt{2}} \frac{1}{\pi} \left[\frac{\pi R_1 \sqrt{\lambda}}{A\sqrt{R_1}} - \frac{\pi A \sqrt{\lambda}}{\lambda \sqrt{R_1}}\right] = \frac{1}{\sqrt{2}} \left[\frac{\sqrt{R_1/\lambda}}{A/\lambda} - \frac{A/\lambda}{\sqrt{R_1/\lambda}}\right]$$

which are (8-105b).

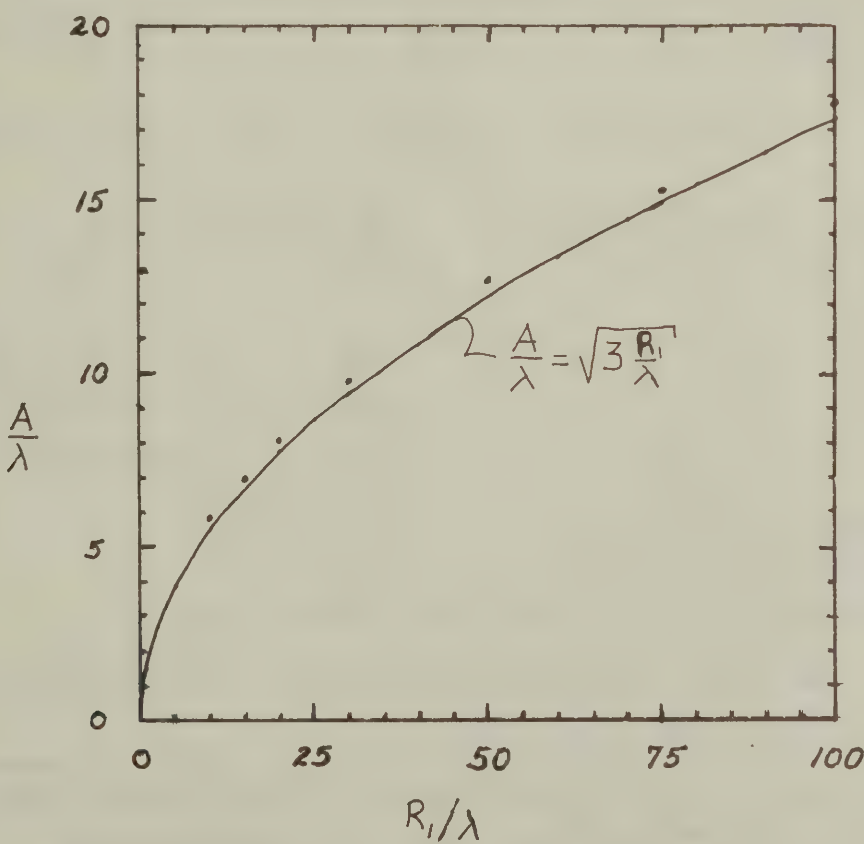
8.4-5

From (8-106)

$A = \sqrt{3\lambda R_1}$

or

$\frac{A}{\lambda} = \sqrt{3\frac{R_1}{\lambda}}$



8.4-6 (a) When the phase error is zero

$$\mathcal{J}_o(\theta,\phi) = \int_{-A/2}^{A/2} \cos\frac{\pi x'}{A}\, e^{j\beta u x'}dx' = \frac{1}{2}\int_{-\frac{A}{2}}^{A/2}\left(e^{j(\frac{\pi}{A}+\beta u)x'} + e^{j(-\frac{\pi}{A}+\beta u)x'}\right)dx'$$

$$= \frac{1}{2}\left\{ \frac{e^{j(\frac{\pi}{A}+\beta u)\frac{A}{2}} - e^{-j(\frac{\pi}{A}+\beta u)\frac{A}{2}}}{j(\frac{\pi}{A}+\beta u)} + \frac{e^{j(-\frac{\pi}{A}+\beta u)\frac{A}{2}} - e^{-j(-\frac{\pi}{A}+\beta u)\frac{A}{2}}}{j(-\frac{\pi}{A}+\beta u)}\right\}$$

$$= \frac{1}{2}\left\{ \frac{e^{j\beta u\frac{A}{2}} + e^{-j\beta u\frac{A}{2}}}{\frac{\pi}{A}+\beta u} + \frac{-e^{j\beta u\frac{A}{2}} - e^{-j\beta u\frac{A}{2}}}{-\frac{\pi}{A}+\beta u}\right\}$$

$$= \frac{\cos\left(\frac{\beta A u}{2}\right)}{\frac{\pi}{A}+\beta u} - \frac{\cos\left(\frac{\beta A u}{2}\right)}{-\frac{\pi}{A}+\beta u} = \cos\left(\frac{\beta A u}{2}\right)\frac{-\frac{\pi}{A}+\beta u-\frac{\pi}{A}-\beta u}{-\left(\frac{\pi}{A}\right)^2+(\beta u)^2}$$

$$= \cos\left(\frac{\beta A u}{2}\right)\frac{-2\frac{\pi}{A}}{-\left(\frac{\pi}{A}\right)^2+(\beta u)^2} = 2\frac{A}{\pi}\;\frac{\cos\left(\frac{\beta A}{2}u\right)}{1-\left(\frac{A}{\pi}\beta u\right)^2}$$

(b)

$$\frac{\mathcal{J}(\theta=0°,\phi=0°)}{\mathcal{J}_o(\theta=0°,\phi=0°)} = \frac{\frac{1}{2}\sqrt{\frac{\pi R_1}{\beta}}\, I(\theta=0°,\phi=0°)}{2\frac{A}{\pi}} = \frac{1}{4}\frac{\pi}{A}\sqrt{\frac{R_1\lambda}{2}}\, I(\theta=0°,\phi=0°)$$

$$= \frac{\pi}{16}\frac{\sqrt{8\lambda R_1}}{A} I(\theta=0°,\phi=0°) = \frac{\pi}{16\sqrt{t}} I(\theta=0°,\phi=0°)$$

(c)

t	$\lvert I(\theta=0°,\phi=0°,t)\rvert$	$\frac{\pi}{16\sqrt{t}}\lvert I(\theta=0°,\phi=0°,t)\rvert$	
$\frac{1}{8}$	1.778	0.9874	−0.11 dB
$\frac{1}{4}$	2.417	0.94915	−0.453 dB
$3/8$	2.777	0.8904	−1.008 dB
$1/2$	2.945	0.8178	−1.747 dB

which can be located on Fig. 8-11

Sample calculation

From Prob. 8.4-4

$$|I(\theta=0°,\phi=0°)| = 2\left[[C(\rho_1)-C(\rho_2)]^2+[S(\rho_1)-S(\rho_2)]^2\right]^{1/2}$$

$$\rho_1 = \frac{1}{\sqrt{2}}\frac{\sqrt{R_1/\lambda}}{A/\lambda} + \frac{1}{\sqrt{2}}\frac{A/\lambda}{\sqrt{R_1/\lambda}} = \frac{1}{4\sqrt{t}} + 2\sqrt{t} \quad \text{since } t=\frac{1}{8}\left(\frac{A}{\lambda}\right)^2\frac{1}{R_1/\lambda},\ \frac{\sqrt{R_1/\lambda}}{A/\lambda} = \frac{1}{\sqrt{8}\sqrt{t}}$$

$$\rho_2 = \frac{1}{4\sqrt{t}} - 2\sqrt{t}$$

For $t = 1/8$ $\rho_1 = 1.414$, $\rho_2 = 0$ and $|I| = 2\sqrt{C^2(1.414)+S^2(1.414)}$

$|I| = 2\sqrt{(0.53)^2+(0.7137)^2} = 1.778$, $\frac{\pi}{16\sqrt{t}}|I| = 0.9874 = -0.11\,dB$

8.4-7 (a) $R_1/\lambda = 5 \qquad \alpha_H = 12.6°$

From geometry

$$\frac{R_1}{\lambda} = \frac{l_H}{\lambda}\cos\alpha_H \Rightarrow \frac{l_H}{\lambda} = \frac{R_1/\lambda}{\cos\alpha_H} = \frac{5}{\cos 12.6°} = 5.123$$

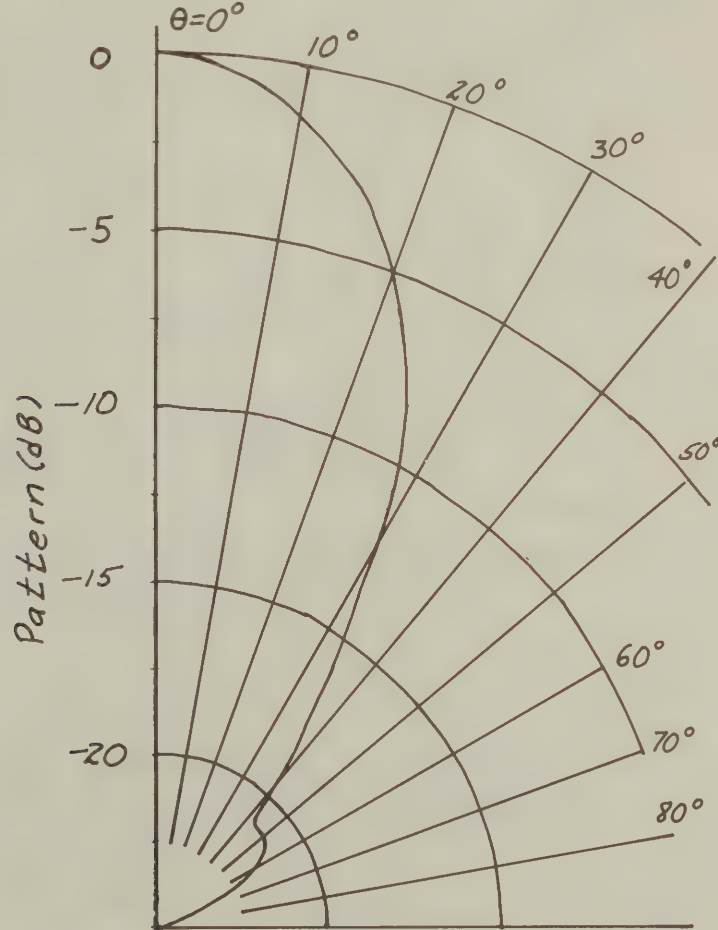
and $$\frac{A}{\lambda} = 2\sqrt{\left(\frac{l_H}{\lambda}\right)^2 - \left(\frac{R_1}{\lambda}\right)^2} = 2\sqrt{(5.12)^2 - (5)^2} = 2.235$$

Then
$$t = \frac{1}{8}\left(\frac{A}{\lambda}\right)^2 \frac{1}{(R_1/\lambda)} = \frac{1}{8}(2.235)^2\frac{1}{5} = \frac{1}{8}$$

The pattern values from Fig. 8-11 with $t=1/8$ are

θ	$\frac{A}{\lambda}\sin\theta$	Pattern value
0°	0	0
5°	0.195	-0.3 dB
10°	0.388	-1.5
15°	0.578	-2.8
20°	0.7644	-5.3
25°	0.9445	-8.0
30°	1.1175	-12.4
35°	1.282	-16.6
40°	1.437	-20.5
45°	1.58	-21.0
50°	1.712	-20.9

θ	$\frac{A}{\lambda}\sin\theta$	Pattern value
55°	1.83	-21.3 dB
60°	1.936	-21.8
65°	2.026	-23.0
70°	2.100	-24.2
75°	2.159	-25.3
80°	2.201	-26.15
85°	2.226	-27.1
90°	2.235	-27.2



8.4-7(con't) (b) From (8-105)

$$D_H = \frac{4\pi b R_1}{\lambda A}\left\{[C(p_1)-C(p_2)]^2+[S(p_1)-S(p_2)]^2\right\}$$

$$p_1 = \frac{1}{\sqrt{2}}\left[\frac{\sqrt{R_1/\lambda}}{A/\lambda}+\frac{A/\lambda}{\sqrt{R_1/\lambda}}\right] = \frac{1}{\sqrt{2}}\left[\frac{\sqrt{5}}{2.235}+\frac{2.235}{\sqrt{5}}\right] = \frac{2}{\sqrt{2}} = 1.414 \text{ and } p_2 = 0$$

$C(p_1) = 0.529$ $\quad S(p_1) = 0.7138$ $\quad C(0) = 0$ $\quad S(0) = 0$

$$D_H = \frac{4\pi b R_1}{\lambda A}\left[(0.529)^2+(0.7138)^2\right] = \frac{4\pi b R_1}{\lambda A}\,0.78935$$

$$\frac{\lambda}{b}D_H = \frac{R_1/\lambda}{A/\lambda}4\pi(0.78935) = \frac{5}{2.235}4\pi(0.78935) = \boxed{22.19}$$

From Fig. 8-12 for $R_1 = 5\lambda$ and $A/\lambda = 2.235$, $\boxed{\frac{\lambda}{b}D_H = 22}$

(c) From (8-56)

$$D_H = \frac{32}{\pi}\frac{Ab}{\lambda^2} \qquad \frac{\lambda}{b}D_H = \frac{32}{\pi}\frac{A}{\lambda} = \frac{32}{\pi}2.235 = \boxed{22.76}$$

which is a close approximation for this small aperture.

8.4-8

(a) WR90 waveguide has $a = 0.9''$ $\quad b = 0.4'' = 1.016$ cm

$\lambda = 3$ cm @ 10 GHz

$D_H = 10^{1.215} = 16.41$ $\quad$ for $G = 12.15$ dB

$$\frac{\lambda}{b}D_H = \frac{3}{1.016}16.41 = 48.4$$

From Fig. 8-12 the optimum occurs for

$R_1/\lambda = 12$ and $A/\lambda = 6$

This is consistent with (8-106):

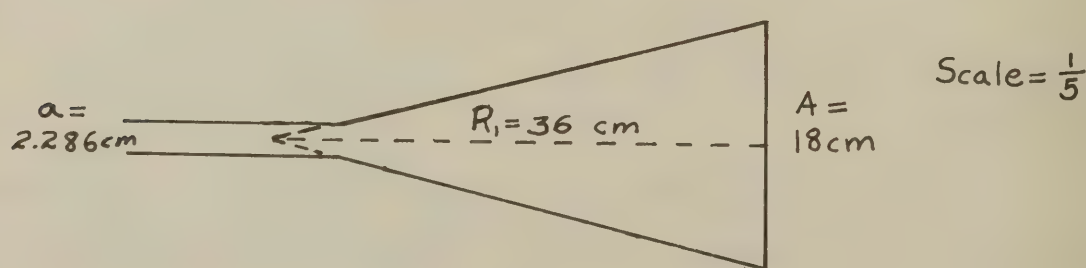
$$A/\lambda = \sqrt{3\,R_1/\lambda} \longrightarrow 6 = \sqrt{3\cdot 12} = \sqrt{36}$$

Hence

$A = 6\lambda = 6\cdot 3 = \boxed{18\text{ cm}}$ $\quad \alpha_H = \tan^{-1}\frac{A/2\lambda}{R_1/\lambda} = \tan^{-1}\frac{3}{12} = \boxed{14°}$

$R_1 = 12\lambda = 12\cdot 3 = \boxed{36\text{ cm}}$

(b)



8.4-8 (con't)

(c) As a check, using (8-105)

$$D_H = \frac{4\pi \frac{b}{\lambda} \frac{R_1}{\lambda}}{A/\lambda} \left\{ \left[C(p_1) - C(p_2)\right]^2 + \left[S(p_1) - S(p_2)\right]^2 \right\}$$

$$p_1 = \frac{1}{\sqrt{2}} \left[\frac{\sqrt{R_1/\lambda}}{A/\lambda} + \frac{A/\lambda}{\sqrt{R_1/\lambda}} \right] = \frac{1}{\sqrt{2}} \left[\frac{\sqrt{12}}{6} + \frac{6}{\sqrt{12}} \right] = 1.633$$

$$p_2 = \frac{1}{\sqrt{2}} \left[\frac{\sqrt{R_1/\lambda}}{A/\lambda} - \frac{A/\lambda}{\sqrt{R_1/\lambda}} \right] = \frac{1}{\sqrt{2}} \left[\frac{\sqrt{12}}{6} - \frac{6}{\sqrt{12}} \right] = -0.8165$$

And

$$C(p_1) = C(1.633) = 0.3467 \; ; \; S(p_1) = S(1.633) = 0.61145$$

$$C(p_2) = -C(0.8165) = -0.7313 \; ; \; S(p_2) = -S(0.8165) = -0.2635$$

So

$$D_H = \frac{4\pi \frac{1.016}{3} 12}{6} \left\{ [0.3467 + 0.7313]^2 + [0.61145 + 0.2635]^2 \right\}$$

$$= 16.407 = \underline{12.15\,\text{dB}} \quad \checkmark$$

8.4-9

Using (8-106), $A = \sqrt{3\lambda R_1}$ in (8-105b)

$$p_{1,2} = \frac{1}{\sqrt{2}} \left[\frac{\sqrt{R_1/\lambda}}{\sqrt{3R_1/\lambda}} \pm \frac{\sqrt{3R_1/\lambda}}{\sqrt{R_1/\lambda}} \right] = \frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{3}} \pm \sqrt{3} \right] = \frac{1}{\sqrt{6}} [1 \pm 3]$$

Or

$$p_1 = \frac{4}{\sqrt{6}} = 1.633 \; ; \; p_2 = \frac{-2}{\sqrt{6}} = -0.8165$$

And (8-106) in (8-105a) is

$$\frac{\lambda}{b} D_H = \frac{4\pi R_1}{\sqrt{3\lambda R_1}} \left\{ \left[C(p_1) - C(p_2)\right]^2 + \left[S(p_1) - S(p_2)\right]^2 \right\}$$

Solving this for R_1/λ

$$\frac{R_1}{\lambda} = 3 \left\{ \frac{\frac{\lambda}{b} D_H}{4\pi\left[[C(p_1) - C(p_2)]^2 + [S(p_1) - S(p_2)]^2 \right]} \right\}^2$$

The values of $\frac{\lambda}{b} D_H = 48.4$ and the Fresnel integrals from Prob. 8.4-8 in the above give

$$\frac{R_1}{\lambda} = 3 \left\{ \frac{48.4}{4\pi\left[[0.3467 + 0.7313]^2 + [0.61145 + 0.2635]^2 \right]} \right\}^2$$

$$= \boxed{11.977} \quad \text{which compares to 12 in Prob. 8.4-8}$$

$$\frac{A}{\lambda} = \sqrt{3\frac{R_1}{\lambda}} = \sqrt{3 \cdot 11.977} = \boxed{5.994} \quad \text{which compares to 6}$$

$R_1 = 11.977\lambda = \underline{35.931\text{ cm}}$ $\quad A = 5.994\lambda = \underline{17.982\text{ cm}}$

8.4-10 Substituting (8-111) into (8-18b)

$$P_y = E_o \underbrace{\int_{-a/2}^{a/2} \cos\frac{\pi x'}{a} e^{j\beta u x'} dx'}_{\mathcal{J}_1} \underbrace{\int_{-B/2}^{B/2} e^{-j\frac{\beta}{2R_2}y'^2} e^{j\beta v y'} dy'}_{\mathcal{J}_2}$$

From (4-25)

$$\mathcal{J}_1 = \frac{2a}{\pi} \frac{\cos\left(\frac{\beta a}{2}u\right)}{1-\left(\frac{2}{\pi}\frac{\beta a}{2}u\right)^2}$$

Completing the square in the second integral

$$\mathcal{J}_2 = e^{j\frac{\beta}{2R_2}(R_2 v)^2}\int_{-B/2}^{B/2} e^{-j\frac{\beta}{2R_2}(y'-R_2 v)^2} dy' = e^{j\frac{\beta}{2R_2}(R_2 v)^2}\int_{r_1}^{r_2} e^{-j\frac{\pi}{2}r^2}\sqrt{\frac{\pi R_2}{\beta}}\,dr$$

where

$$\frac{\pi}{2}r^2 = \frac{\beta}{2R_2}(y'-R_2 v)^2 \quad \text{or} \quad r = \sqrt{\frac{2\beta}{\pi 2R_2}}(y'-R_2 v) \quad \text{and} \quad dy' = \sqrt{\frac{\pi R_2}{\beta}}\,dr$$

and

$$r_1 = \sqrt{\frac{\beta}{\pi R_2}}\left(-\frac{B}{2}-R_2 v\right) \qquad r_2 = \sqrt{\frac{\beta}{\pi R_2}}\left(\frac{B}{2}-R_2 v\right) \qquad \text{(8-112b)}$$

So

$$\mathcal{J}_2 = \sqrt{\frac{\pi R_2}{\beta}} e^{j\frac{\beta}{2R_2}(R_2 v)^2}\left[C(r_2) - jS(r_2) - C(r_1) + jS(r_1)\right]$$

Now from (8-24) with $Q_x = -P_y/\eta$ as in (8-61),

$$\vec{E} = j\beta\frac{e^{-j\beta r}}{4\pi r}\left[\hat{\theta}\left(P_y \sin\phi + \eta\cos\theta\frac{P_y}{\eta}\sin\phi\right) + \hat{\phi}\left(\cos\theta\, P_y\cos\phi + \eta\cos\phi\frac{P_y}{\eta}\right)\right]$$

$$= j\beta\frac{e^{-j\beta r}}{4\pi r}(1+\cos\theta)\left[\hat{\theta}\sin\phi + \hat{\phi}\cos\phi\right]P_y$$

Combining above results for $P_y = E_o \mathcal{J}_1 \mathcal{J}_2$

$$\vec{E} = E_o j\beta\sqrt{\frac{\pi R_2}{\beta}}\frac{4a}{\pi}\frac{e^{-j\beta r}}{4\pi r}e^{j\frac{\beta R_2}{2}v^2}\left[\hat{\theta}\sin\phi + \hat{\phi}\cos\phi\right]\frac{1+\cos\theta}{2}$$

$$\cdot \frac{\cos\left(\frac{\beta a}{2}u\right)}{1-\left(\frac{\beta a}{\pi}u\right)^2}\left[C(r_2) - jS(r_2) - C(r_1) + jS(r_1)\right] \text{ which is (8-112a).}$$

8.4-11 In the E-plane $\phi = 90°$ and $v = \sin\theta\sin\phi = \sin\theta$

Then using (8-112b)

$$r_3 = r_1(\phi = 90°) = \sqrt{\frac{\beta}{\pi R_2}}\left(-\frac{B}{2} - R_2\sin\theta\right) = \sqrt{\frac{2}{R_2/\lambda}}\left(-\frac{1}{2}\frac{B}{\lambda} - \frac{R_2}{\lambda}\sin\theta\right)$$

And using $R_2/\lambda = \frac{1}{8}\left(\frac{B}{\lambda}\right)^2\frac{1}{s}$ from (8-114), so

$$r_3 = \frac{4\sqrt{s}}{B/\lambda}\left(-\frac{1}{2}\frac{B}{\lambda} - \frac{1}{8}\left(\frac{B}{\lambda}\right)^2\frac{1}{s}\sin\theta\right) = 2\sqrt{s}\left[-1 - \frac{1}{4}\frac{1}{s}\left(\frac{B}{\lambda}\sin\theta\right)\right] \quad \text{(8-115b)}$$

8.4-11 (con't)

similarly $\quad \Gamma_4 = \Gamma_2(\phi=90°) = 2\sqrt{5}\left[1-\frac{1}{4}\frac{1}{5}\left(\frac{B}{\lambda}\sin\theta\right)\right] \qquad (8\text{-}115b)$

The E-plane pattern follows from (8-112a) with $\phi=90°$

$$|F_E| \propto \frac{1+\cos\theta}{2}\left\{\left[C(\Gamma_4)-C(\Gamma_3)\right]^2+\left[S(\Gamma_4)-S(\Gamma_3)\right]^2\right\}^{1/2} \text{ in magnitude}$$

At $\theta=0°$ $\quad \Gamma_3=-2\sqrt{5}$ and $\Gamma_4=2\sqrt{5}$ and then the previous equation reduces to

$$(1)\left\{4C^2(2\sqrt{5})+4S^2(2\sqrt{5})\right\}^{1/2}$$

which is the pattern maximum. Using this to normalize

$$|F_E(\theta)| = \frac{1+\cos\theta}{2}\left\{\frac{\left[C(\Gamma_4)-C(\Gamma_3)\right]^2+\left[S(\Gamma_4)-S(\Gamma_3)\right]^2}{4\left[C^2(2\sqrt{5})+S^2(2\sqrt{5})\right]}\right\}^{1/2}$$

8.4-12

From (8-90)

$$\Delta R = R-R_1 = \frac{1}{2}\frac{x^2}{R_1}$$

So

$$\Delta R_{max} = \frac{1}{2}\frac{(A/2)^2}{R_1} = \frac{1}{8}\frac{A^2}{R_1}$$

And from (8-107)

$$t_{op} = \frac{A^2}{8\lambda R_1} = \frac{3}{8} \Rightarrow \frac{A^2}{8R_1} = \frac{3}{8}\lambda$$

Combining $\quad \boxed{\Delta R_{max} = \frac{3}{8}\lambda} \qquad$ H-plane horn

Similarly for the E-plane horn

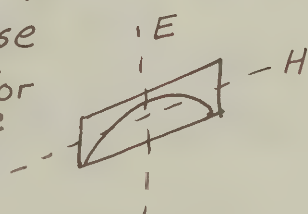
$$\Delta R_{max} = \frac{1}{2}\frac{(B/2)^2}{R_2} = \frac{1}{8}\frac{B^2}{R_2}$$

and

$$s_{op} = \frac{B^2}{8\lambda R_2} = \frac{1}{4} \Rightarrow \frac{B^2}{8R_2} = \frac{1}{4}\lambda$$

$\therefore \quad \boxed{\Delta R_{max} = \frac{1}{4}\lambda} \qquad$ E-plane horn

8.4-13 The phase error parameters for optimum E- and H-plane sectoral horns are different because the amplitude distributions are different for each plane. In the H-plane the amplitude decreases toward the edges; thus larger phase errors may be tolerated due to the reduction from amplitude weighting.



8.4-14 WR90: $a=2.286$ cm $b=1.016$ cm $f=10$ GHz, $\lambda=3$cm

$HP_{E^\circ}=11^\circ$ $G=14.9$ dB

From (8-119) $HP_{E^\circ}=54\frac{\lambda}{B}^\circ \Rightarrow \frac{B}{\lambda}=\frac{54}{HP_{E^\circ}}=\frac{54}{11}=4.91$

So
$$B=4.91\lambda=4.91(3)=14.73 \text{ cm}$$

And $\frac{a}{\lambda}=\frac{2.286}{3}=0.762$

Also $D_E \simeq G_E = 10^{1.49} = 30.9$

Thus
$$\frac{\lambda}{a}D_E=\frac{1}{0.762}30.9=40.55$$

From Fig. 8-15 with $\frac{\lambda}{a}D_E=40.55$ and $\frac{B}{\lambda}=4.91\simeq 5$

$R_2/\lambda=12$ or from (8-117) $R_2/\lambda=\frac{1}{2}\left(\frac{B}{\lambda}\right)^2=12.05$

Finally from (8-109a)
$$\ell_E=\sqrt{R_2^2+\left(\frac{B}{2}\right)^2}=\sqrt{(36)^2+\left(\frac{14.73}{2}\right)^2}=\boxed{36.74 \text{ cm}}$$

since $R_2=12\lambda=12(3)=36$ cm

$$R_E=(B-b)\sqrt{\left(\frac{\ell_E}{B}\right)^2-\frac{1}{4}}=(14.73-1.016)\sqrt{\left(\frac{36.74}{14.73}\right)^2-\frac{1}{4}}$$
$$=\boxed{33.51 \text{ cm}} \quad B=4.91\lambda=4.91(3)=\boxed{14.73 \text{ cm}}$$

8.4-15 (E-plane sectoral horn of Jull & Allen [6])

WR284: $a=2.84''=7.2136$ cm, $b=1.34''=3.4036$ cm

$A=a=7.2136$ cm $B=24.0$ cm $\alpha_E=16.5^\circ$

At $f=3.75$ GHz ($\lambda=8$ cm) $G_E=14.5$ dB

From (8-109b)
$$R_2=\frac{B}{2\tan\alpha_E}=\frac{24}{2\tan 16.5^\circ}=40.511 \text{ cm}$$

From (8-109a)
$$\ell_E=\sqrt{R_2^2+\left(\frac{B}{2}\right)^2}=42.251 \text{ cm}$$

(a)
$$D_E=\frac{32}{\pi}\frac{aB}{\lambda^2}\frac{C^2(q)+S^2(q)}{q^2} \qquad (8\text{-}116)$$

where
$$q=B/\sqrt{2\lambda R_2}=24/\sqrt{2(8)(40.511)}=0.94268$$

$C(0.94268)=0.774845$ $S(0.94268)=0.381233$

8.4-15a (con't)

So
$$D_E = \frac{32}{\pi}\,\frac{(7.2136)(24)}{8^2}\,\frac{(0.774845)^2+(0.381233)^2}{(0.94268)^2}$$
$$= 23.122 = \boxed{13.640\text{ dB}}$$

(b)
$$\lambda_g = \frac{\lambda}{\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}} = \frac{8}{\sqrt{1-\left(\frac{8}{2\cdot 7.2136}\right)^2}} = 9.6133 \text{ cm}$$
$$e^{\pi\frac{a}{\lambda}\left(1-\frac{\lambda}{\lambda_g}\right)} = e^{\pi\frac{7.2136}{8}\left(1-\frac{8}{9.6133}\right)} = e^{0.4754} = 1.60866$$
$$G_E = \frac{16\,a\,B}{\lambda^2(1+\lambda_g/\lambda)}\,\frac{C^2(g)+S^2(g)}{g^2}\,e^{\pi\frac{a}{\lambda}\left(1-\frac{\lambda}{\lambda_g}\right)} \qquad (8\text{-}120)$$
$$= \frac{16(7.2136)(24)}{8^2(1+9.6133/8)}\,\frac{(0.774845)^2+(0.381233)^2}{(0.94268)^2}$$
$$= 26.538 = \boxed{14.2386\text{ dB}}$$

8.4-16 The radiation fields of the E and H plane sectoral horns are not zero for $\theta > \pi/2$ as we claimed in connection with (8-24) because of the imposed plane wave behavior over the aperture fields; see (8-61). Only when the exact aperture fields are used is the result zero for $\theta > \pi/2$ using (8-24).

8.4-17 Using (8-86) and (8-110) in (8-123)
$$R_E = (B-b)\sqrt{\left(\frac{\ell_E}{B}\right)^2-\frac{1}{4}} = (A-a)\sqrt{\left(\frac{\ell_H}{A}\right)^2-\frac{1}{4}} = R_H$$
Inserting
$$A = \sqrt{3\lambda R_1} \approx \sqrt{3\lambda\ell_H} \qquad B = \sqrt{2\lambda R_2} \approx \sqrt{2\lambda\ell_E} \qquad (8\text{-}124)$$
and squaring gives
$$\left(\sqrt{2\lambda\ell_E}-b\right)^2\left(\frac{\ell_E^2}{2\lambda\ell_E}-\frac{1}{4}\right) = \left(\sqrt{3\lambda\ell_H}-a\right)^2\left(\frac{\ell_H^2}{3\lambda\ell_H}-\frac{1}{4}\right)$$
or
$$\left(\sqrt{2\frac{\ell_E}{\lambda}}-\frac{b}{\lambda}\right)^2\left(2\frac{\ell_E}{\lambda}-1\right) = \left(\sqrt{3\frac{\ell_H}{\lambda}}-\frac{a}{\lambda}\right)^2\left(\frac{4}{3}\frac{\ell_H}{\lambda}-1\right) \qquad (*)$$
Working from (8-125)
$$G = \frac{4\pi}{\lambda^2}\,\frac{AB}{2} = \frac{4\pi}{\lambda^2}\,\frac{\sqrt{3\lambda\ell_H}\sqrt{2\lambda\ell_E}}{2} = 2\pi\sqrt{6}\sqrt{\frac{\ell_H}{\lambda}\frac{\ell_E}{\lambda}}$$

8.4-17 (con't)

So $\sqrt{\frac{\ell_H}{\lambda}} = \frac{G}{2\pi\sqrt{6}}\sqrt{\frac{\lambda}{\ell_E}} = \frac{G}{2\pi\sqrt{6}}\frac{1}{\sqrt{\sigma}}$ where $\sigma = \frac{\ell_E}{\lambda}$

Using this in (*)

$$\left(\sqrt{2}\sqrt{\sigma} - \frac{b}{\lambda}\right)^2(2\sigma-1) = \left(\frac{G}{2\sqrt{2}\,\pi}\frac{1}{\sqrt{\sigma}} - \frac{a}{\lambda}\right)^2\left(\frac{G^2}{18\pi^2}\frac{1}{\sigma} - 1\right)$$

which is (8-126).

<u>8.4-18</u> (Example 8-7 ; SA Model SA 12- 8.2)

$$\ell_E = \sigma\lambda = 10.17(3.226) = \underline{32.81 \text{ cm}}$$

$$B \approx \sqrt{2\lambda\ell_E} = \sqrt{2(3.226)(32.81)} = \underline{14.55 \text{ cm}}$$

$$A = \frac{\lambda^2}{2\pi}\frac{G}{B} = \frac{(3.226)^2}{2\pi}\frac{162.18}{14.55} = \underline{18.46 \text{ cm}}$$

$$\ell_H \approx \frac{A^2}{3\lambda} = \frac{(18.46)^2}{3(3.226)} = \underline{35.21 \text{ cm}}$$

$$R_E = (B-b)\sqrt{\left(\frac{\ell_E}{B}\right)^2 - \frac{1}{4}} = (14.55-1.016)\sqrt{\left(\frac{32.81}{14.55}\right)^2 - \frac{1}{4}} = \underline{29.76 \text{ cm}}$$

$$R_H = (A-a)\sqrt{\left(\frac{\ell_H}{A}\right)^2 - \frac{1}{4}} = (18.46-2.286)\sqrt{\left(\frac{35.21}{18.46}\right)^2 - \frac{1}{4}} = \underline{29.77 \text{ cm}}$$

<u>8.4-19</u>

$a = 2.286$ cm $\quad b = 1.016$ cm $\quad \lambda = 3$ cm @ 10 GHz

$\frac{a}{\lambda} = 0.762 \quad \frac{b}{\lambda} = 0.3387$

$G = 10^2 = 100 \quad \frac{G}{2\sqrt{2}\,\pi} = 11.254 \quad \frac{G^2}{18\pi^2} = 56.29$

So (8-126) is

$$\left(\sqrt{2\sigma} - 0.3387\right)^2(2\sigma-1) = \left(11.254\frac{1}{\sqrt{\sigma}} - 0.762\right)^2\left(56.29\frac{1}{\sigma} - 1\right)$$

By trial and error starting with $\sigma_1 = G/2\pi\sqrt{6} = 6.50$, the solution is $\sigma = 6.183$.

Then

$$\ell_E = \sigma\lambda = (6.183)(3) = \underline{18.55 \text{ cm}}$$

$$B \approx \sqrt{2\lambda\ell_E} = \sqrt{2(3)(18.55)} = \underline{10.55 \text{ cm}}$$

$$A = \frac{\lambda^2}{2\pi}\frac{G}{B} = \frac{(3)^2}{2\pi}\frac{100}{10.55} = \underline{13.58 \text{ cm}}$$

8.4-19 (con't)

$$\ell_H \approx \frac{A^2}{3\lambda} = \frac{(13.58)^2}{3(3)} = \underline{20.48\text{ cm}}$$

$$R_H = (A-a)\sqrt{\left(\frac{\ell_H}{A}\right)^2 - \frac{1}{4}} = (13.58-2.286)\sqrt{\left(\frac{20.48}{13.58}\right)^2 - \frac{1}{4}} = \underline{16.07\text{ cm}}$$

$$R_E = (B-b)\sqrt{\left(\frac{\ell_E}{B}\right)^2 - \frac{1}{4}} = (10.55-1.016)\sqrt{\left(\frac{18.55}{10.55}\right)^2 - \frac{1}{4}} = \underline{16.07\text{ cm}}$$

As a gain check

$$\frac{A}{\lambda} = \frac{13.58}{3} = 4.53 \text{ and } \frac{R_1}{\lambda} \approx \frac{\ell_H}{\lambda} = \frac{20.48}{3} = 6.83$$

in Fig. 8-12 give $\frac{\lambda}{B} D_H = 35.$

and

$$\frac{B}{\lambda} = \frac{10.55}{3} = 3.52 \text{ and } \frac{R_2}{\lambda} \approx \frac{\ell_E}{\lambda} = \frac{18.55}{3} = 6.18$$

in Fig. 8-15 give $\frac{\lambda}{A} D_E = 29$

Then, from (8-122)

$$D_p = \frac{\pi}{32}\left(\frac{\lambda}{A}D_E\right)\left(\frac{\lambda}{B}D_H\right) = \frac{\pi}{32}(29)(35) = 99.64 = \underline{20.0\text{dB}} \checkmark$$

8.4-20

$G = 24.7\text{dB}$ @ 24 GHz ($\lambda = 1.25$cm) $\quad G = 10^{2.47} = 295.12$

WR42: $a = 0.42'' = 1.0668$ cm $b = 0.17'' = 0.4318$ cm

$a/\lambda = 0.85344 \qquad b/\lambda = 0.34544$

(a)

$$\frac{G}{2\sqrt{2}\,\pi} = 33.213 \qquad \frac{G^2}{18\pi^2} = 490.258$$

So (8-126) is

$$(\sqrt{2\sigma} - 0.34544)^2(2\sigma - 1) = \left(33.213\frac{1}{\sqrt{\sigma}} - 0.85394\right)^2\left(\frac{490.258}{\sigma} - 1\right)$$

Starting with $\sigma_1 = G/2\pi\sqrt{6} = 19.175$ and iterating yields the solution $\sigma = 18.565 = \ell_E/\lambda$

Then

$$\frac{B}{\lambda} \approx \sqrt{2\frac{\ell_E}{\lambda}} = \sqrt{2(18.565)} = 6.0934, \quad B = 6.0934(1.25) = \boxed{7.617\text{ cm}}$$

$$\frac{A}{\lambda} = \frac{G}{2\pi}\frac{1}{B/\lambda} = 7.708, \qquad A = 7.708(1.25) = \boxed{9.635\text{ cm}}$$

$$\frac{\ell_H}{\lambda} \approx \frac{1}{3}\left(\frac{A}{\lambda}\right)^2 = 19.804 \qquad \ell_H = 19.804(1.25) = 24.755\text{ cm}$$

8.4-20 (con't)

$$\frac{R_H}{\lambda}=\left(\frac{A}{\lambda}-\frac{a}{\lambda}\right)\sqrt{\left(\frac{\ell_H/\lambda}{A/\lambda}\right)^2-\frac{1}{4}}=(7.708-0.85344)\sqrt{\left(\frac{19.804}{7.708}\right)^2-\frac{1}{4}}=17.275$$

$$R_H = 17.275(1.25) = \boxed{21.59\text{ cm}}$$

$$\frac{\ell_E}{\lambda}=18.565 \qquad \ell_E = 18.565(1.25) = 23.21\text{ cm}$$

$$\frac{R_E}{\lambda}=\left(\frac{B}{\lambda}-\frac{b}{\lambda}\right)\sqrt{\left(\frac{\ell_E/\lambda}{B/\lambda}\right)^2-\frac{1}{4}}=(6.0934-0.34544)\sqrt{\left(\frac{18.565}{6.0934}\right)^2-\frac{1}{4}}$$

$$=17.275 \qquad R_E = 17.275(1.25) = \boxed{21.59\text{cm}}$$

Note $R_H = R_E$ ✓

(b)

From (8-108) $HP_H = 78\frac{\lambda}{A}^\circ = \frac{78}{7.708} = \boxed{10.1^\circ}$

From (8-119) $HP_E = 54\frac{\lambda}{B}^\circ = \frac{54}{6.0934} = \boxed{8.86^\circ}$

(c)

From (8-82)

$$G = \frac{26{,}000}{HP_{E^\circ}\,HP_{H^\circ}} = \frac{26{,}000}{(8.86)(10.1)} = 290.54 = \boxed{24.6\text{ dB}}$$

(d)

From (4-28) for a cosine tapered line source

$$HP_H = 68.2\frac{\lambda}{A}^\circ = \frac{68.2}{7.708} = \boxed{8.85^\circ}$$

From (4-14) for a uniform amplitude line source

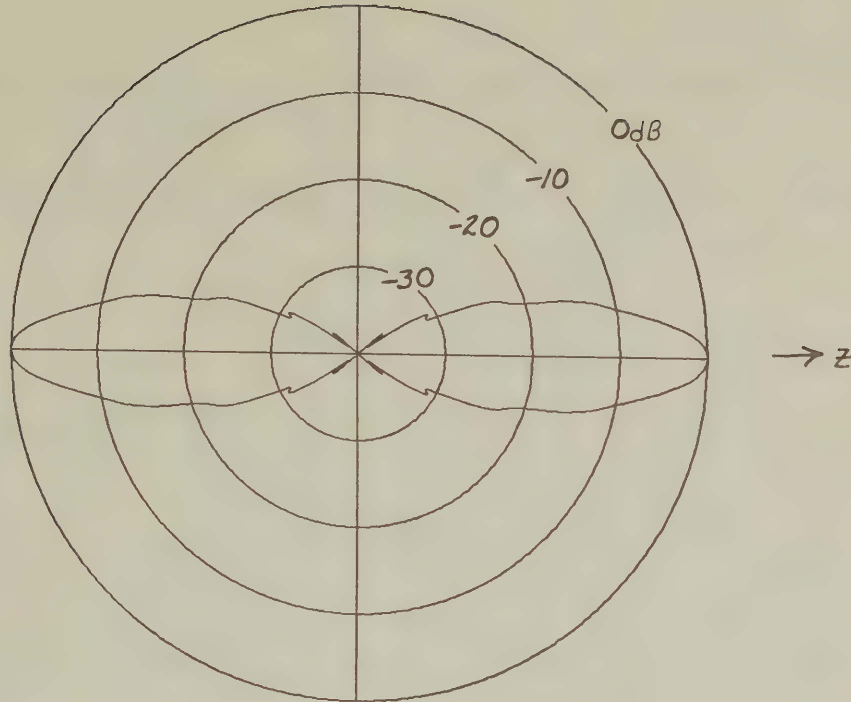
$$HP_E = 50.8\frac{\lambda}{B}^\circ = \frac{50.8}{6.0934} = \boxed{8.34^\circ}$$

And we see that the actual HP's of (b) are larger than those computed here for zero phase error because of the deleterious effects of the quadratic phase error.

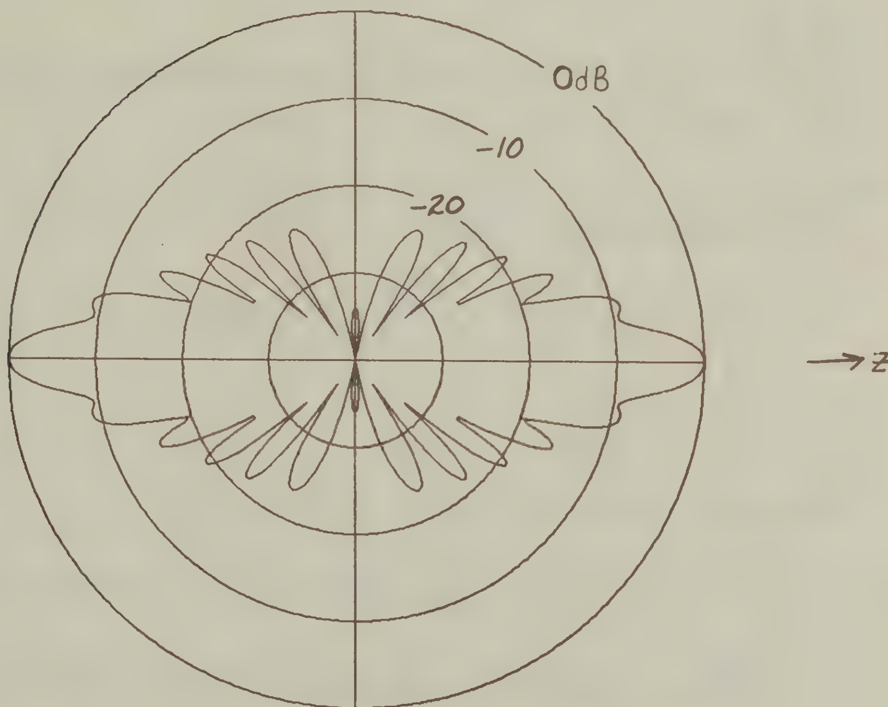
8.4-20 (con't)

(e)

H-plane pattern



E-plane pattern



8.4-21

The criterion for realizability of a pyramidal horn antenna is

$$R_E = R_H = R_p \qquad (8\text{-}123)$$

Substituting (8-86) and squaring gives

$$(B-b)^2\left[\left(\frac{\ell_E}{B}\right)^2-\frac{1}{4}\right]=(A-a)^2\left[\left(\frac{\ell_H}{A}\right)^2-\frac{1}{4}\right]$$

But from (8-109a) and (8-85a)

$$\left(\frac{\ell_E}{B}\right)^2=\frac{1}{4}+\left(\frac{R_2}{B}\right)^2 \qquad \left(\frac{\ell_H}{A}\right)^2=\frac{1}{4}+\left(\frac{R_1}{A}\right)^2$$

So

$$(B-b)^2\left(\frac{R_2}{B}\right)^2=(A-a)^2\left(\frac{R_1}{A}\right)^2$$

Or

$$\frac{R_1}{A}=\frac{B-b}{A-a}\,\frac{R_2}{B} \quad \text{or} \quad R_1=\frac{1-\frac{b}{B}}{1-\frac{a}{A}}R_2 \quad \text{or} \quad \boxed{\frac{R_1}{\lambda}=\frac{1-\frac{b/\lambda}{B/\lambda}}{1-\frac{a/\lambda}{A/\lambda}}\,\frac{R_2}{\lambda}}$$

8.4-22

$\lambda = 3.75\text{ cm} @ 8\text{ GHz}$ WR90: $a=2.286\text{ cm}, b=1.016\text{ cm}, \frac{a}{\lambda}=0.6096, \frac{b}{\lambda}=0.2709$

$HP_{E^\circ}=HP_{H^\circ}=12^\circ$ or $\theta_E=\theta_H=6^\circ$

(a)

For optimum operation, from above (8-108) and (8-119),

$$\frac{A}{\lambda}=\frac{0.68}{\sin\theta_H}=\frac{0.68}{\sin 6^\circ}=\underline{6.5054}$$

$$\frac{B}{\lambda}=\frac{0.47}{\sin\theta_E}=\frac{0.47}{\sin 6^\circ}=\underline{4.496}$$

(b)

Optimum pyramidal horn results yield

$$\frac{R_{1op}}{\lambda}=\frac{1}{t}\,\frac{1}{8}\left(\frac{A}{\lambda}\right)^2=\frac{1}{3/8}\,\frac{1}{8}(6.5054)^2=14.107$$

$$\frac{R_{2op}}{\lambda}=\frac{1}{s}\,\frac{1}{8}\left(\frac{B}{\lambda}\right)^2=\frac{1}{1/4}\,\frac{1}{8}(4.496)^2=10.107$$

And

$$\frac{1-\frac{b}{B}}{1-\frac{a}{A}}=\frac{1-\frac{b/\lambda}{B/\lambda}}{1-\frac{a/\lambda}{A/\lambda}}=1.03692$$

From Prob. 8.4-21

$$\frac{R_1}{\lambda}=\frac{1-\frac{b}{B}}{1-a/A}\,\frac{R_2}{\lambda} \quad \text{or} \quad \underbrace{\frac{R_{1op}}{\lambda}f}_{R_1/\lambda}=\frac{1-\frac{b}{B}}{1-\frac{a}{A}}\,\underbrace{\frac{R_{2op}}{\lambda}\,\frac{1}{f}}_{R_2/\lambda}$$

8.4-22 (con't)

So

$$14.107 f = 1.03692(10.107)\frac{1}{f}$$

$$f = \sqrt{\frac{1.03692(10.107)}{14.107}} = \boxed{0.8619}$$

and

$$\frac{R_1}{\lambda} = \frac{R_{1op}}{\lambda} f = 14.107(0.8619) = 12.159$$

$$\frac{R_2}{\lambda} = \frac{R_{2op}}{\lambda}\frac{1}{f} = \frac{10.107}{0.8619} = 11.726$$

As a check

$$\frac{\ell_H}{\lambda} = \sqrt{\left(\frac{A}{2\lambda}\right)^2 + \left(\frac{R_1}{\lambda}\right)^2} = 12.5865 \qquad \frac{R_H}{\lambda} = \left(\frac{A}{\lambda} - \frac{a}{\lambda}\right)\sqrt{\frac{(\ell_H/\lambda)^2}{(A/\lambda)^2} - \frac{1}{4}} = 11.0196$$

$$\frac{\ell_E}{\lambda} = \sqrt{\left(\frac{B}{2\lambda}\right)^2 + \left(\frac{R_2}{\lambda}\right)^2} = 11.9397 \qquad \frac{R_E}{\lambda} = \left(\frac{B}{\lambda} - \frac{b}{\lambda}\right)\sqrt{\frac{(\ell_E/\lambda)^2}{(B/\lambda)^2} - \frac{1}{4}} = 11.0196$$

(c)

$$t = \frac{1}{8}\left(\frac{A}{\lambda}\right)^2 \frac{1}{R_1/\lambda} = \frac{1}{8}(6.5054)^2\frac{1}{12.159} = \underline{0.43507}$$ 16% increase over 3/8

$$s = \frac{1}{8}\left(\frac{B}{\lambda}\right)^2 \frac{1}{R_2/\lambda} = \frac{1}{8}(4.496)^2\frac{1}{11.726} = \underline{0.2155}$$ 14% decrease from 1/4

(d)

$A = (6.5054)3.75 = \underline{24.395 \text{ cm}}$

$B = (4.496)3.75 = \underline{16.86 \text{ cm}}$

$\ell_H = (12.5865)3.75 = \underline{47.20 \text{ cm}}$

$\ell_E = (11.9397)3.75 = \underline{44.77 \text{ cm}}$

$R_H = R_E = (11.0196)\,3.75 = \underline{41.32 \text{ cm}}$

(e)

From Fig. 8-12 with $\frac{A}{\lambda} = 6.5$ and $\frac{R_1}{\lambda} \approx 12.2$ $\qquad \frac{\lambda}{B} D_H \approx 47.5$

From Fig. 8-15 with $\frac{B}{\lambda} \approx 4.5$ and $\frac{R_2}{\lambda} \approx 11.7$ $\qquad \frac{\lambda}{A} D_E \approx 40$

So $G \approx D_p = \frac{\pi}{32}\left(\frac{\lambda}{A}D_E\right)\left(\frac{\lambda}{B}D_H\right) = \frac{\pi}{32}(40)(47.5) = 186.532 = \boxed{22.7 \text{ dB}}$

(f)

$D_u = 4\pi\left(\frac{A}{\lambda}\right)\left(\frac{B}{\lambda}\right) = 4\pi(6.5054)(4.496) = 367.54\,;\ \epsilon_{ap} = \frac{G}{D_u} = \frac{186.53}{367.54} = \boxed{50.75\%}$

8.5-1

Assume a plane wave behavior of the aperture fields:

$$\vec{E}_a = \hat{x} E_o \qquad \vec{H}_a = \hat{y}\frac{E_o}{\eta}$$

Then— see (8-16) and (8-17) —

$$\vec{P} = \hat{x} P_x \qquad \vec{Q} = \hat{y}\frac{P_x}{\eta} \qquad P_y = 0,\ Q_x = 0$$

Thus the electric and magnetic current formulation of (8-24) reduces to

$$\vec{E} = j\beta\frac{e^{-j\beta r}}{4\pi r}\left\{\hat{\theta}\left[P_x\cos\phi + \eta\cos\theta\frac{P_x}{\eta}\cos\phi\right] + \hat{\phi}\left[-P_x\cos\theta\sin\phi - \eta\frac{P_x}{\eta}\sin\phi\right]\right\}$$

$$= \underline{[\hat{\theta}\cos\phi - \hat{\phi}\sin\phi](1+\cos\theta)P_x\, j\beta\frac{e^{-j\beta r}}{4\pi r}}$$

8.5-2

The uniform circular aperture pattern is

$$f(\theta) = \frac{2J_1(\beta a\sin\theta)}{\beta a\sin\theta} \qquad (8\text{-}137)$$

From (F-7)

$$J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m\, x^{2m+n}}{m!(m+n)!\,2^{2m+n}}$$

Retaining only the first term (since $x \ll 1$ for θ small) for $n=1$

$$J_1(x) \approx \frac{(1)x^1}{(1)(1)2} = \frac{x}{2} \qquad \text{for } x \ll 1$$

So

$$\lim_{\theta\to 0}\frac{2J_1(\beta a\sin\theta)}{\beta a\sin\theta} = \frac{2\,\frac{\beta a\sin\theta}{2}}{\beta a\sin\theta} = \boxed{1}$$

8.5-3

From (8-141)

$$f_{un} = 2\pi\int_0^a\left\{C + (1-C)\left[1-\left(\frac{r'}{a}\right)^2\right]^n\right\} r' J_o(\beta r'\sin\theta)\,dr'$$

Let $x = r'/a$ and $b = \beta a\sin\theta$

Then

$$f_{un} = 2\pi\int_0^1\left\{C + (1-C)[1-x^2]^2\right\} ax\, J_o(bx)\, a\,dx$$

$$= 2\pi a^2\left\{C\int_0^1 x J_o(bx)\,dx + (1-C)\int_0^1(1-x^2)^n x J_o(bx)\,dx\right\}$$

8.5-3 (con't)

Using (8-134) $\int u J_0(u)\,du = u J_1(u)$ we see that

$$\int_0^1 x J_0(bx)\,dx = \int_0^b \frac{y}{b} J_0(y) \frac{dy}{b} \quad \text{where } y = bx$$

$$= \frac{1}{b^2}\int_0^b y J_0(y)\,dy = \frac{1}{b^2}\left[y J_1(y)\right]_0^b = \frac{J_1(b)}{b}$$

And

$$\int_0^1 (1-x^2)^n x J_0(bx)\,dx = \frac{2^n n!}{b^{n+1}} J_{n+1}(b) \quad \text{from (8-143)}$$

So

$$f_{un} = 2\pi a^2\left\{C\frac{J_1(b)}{b} + (1-C)\frac{2^n n!}{b^{n+1}} J_{n+1}(b)\right\}$$

$$= \pi a^2\left\{C\frac{2J_1(\beta a\sin\theta)}{\beta a\sin\theta} + \frac{1-C}{n+1}\,\frac{2^{n+1}(n+1)!\,J_{n+1}(\beta a\sin\theta)}{(\beta a\sin\theta)^{n+1}}\right\}$$

$$= \pi a^2\left[C f(\theta, n=0) + \frac{1-C}{n+1} f(\theta, n)\right] \quad \text{see (8-145)}$$

But

$$f_{un}(\theta=0) = f_{un\,max} = \pi a^2\left[C + \frac{1-C}{n+1}\right]$$

Thus

$$f(\theta, n, C) = \frac{f_{un}}{f_{un\,max}} = \boxed{\frac{C f(\theta, n=0) + \frac{1-C}{n+1} f(\theta, n)}{C + \frac{1-C}{n+1}}}$$

8.5-4

$$\epsilon_t = \frac{D}{D_u} = \frac{\dfrac{4\pi}{\lambda^2}\dfrac{\left|\iint_S \vec{E}_a\,ds'\right|^2}{\iint_S |\vec{E}_a|^2 ds'}}{\dfrac{4\pi}{\lambda^2} A_p} = \frac{1}{A_p}\frac{\left|\iint \vec{E}_a\,ds'\right|^2}{\iint |\vec{E}_a|^2 ds'} = \frac{1}{A_p}\,\frac{\vartheta_1}{\vartheta_2}$$

$$|\vec{E}_a| = C + (1-C)\left[1 - \left(\tfrac{r}{a}\right)^2\right]^n$$

(a) $n = 1$

$$\vartheta_1 = \left\{2\pi\int_0^a \left[C + (1-C)\left(1-\left(\tfrac{r}{a}\right)^2\right)\right] r\,dr\right\}^2$$

$$= \left\{2\pi\int_0^a \left[r - (1-C)\frac{1}{a^2} r^3\right] dr\right\}^2 = \left\{2\pi\left[\frac{r^2}{2} - (1-C)\frac{1}{a^2}\frac{r^4}{4}\right]_0^a\right\}^2$$

$$= \left\{\pi a^2\left[1 - (1-C)\tfrac{1}{2}\right]\right\}^2 = \left\{\pi a^2\left(\tfrac{1}{2} + \tfrac{C}{2}\right)\right\}^2$$

8.5-4 (con't)

$$\mathcal{J}_2 = \int_0^{2\pi}\int_0^a \left\{ C^2 + (1-C)^2\left[1-\left(\tfrac{r}{a}\right)^2\right]^2 + 2C(1-C)\left[1-\left(\tfrac{r}{a}\right)^2\right]\right\} r\,dr\,d\phi$$

$$= 2\pi\int_0^a \left\{ C^2 + (1-C)^2\left[1-\frac{2}{a^2}r^2+\frac{1}{a^4}r^4\right] + 2C(1-C)\left(1-\frac{1}{a^2}r^2\right)\right\} r\,dr$$

$$= 2\pi\int_0^a \left\{ r + \frac{2}{a^2}(C-1)r^3 + \frac{1}{a^4}(C-1)^2 r^5\right\} dr$$

after collection of like terms in r

$$= 2\pi\left[\frac{r^2}{2}+\frac{2}{a^2}(C-1)\frac{r^4}{4}+\frac{1}{a^4}(C-1)^2\frac{r^6}{6}\right]_0^a$$

$$= 2\pi a^2\left[\frac{1}{2}+\frac{1}{2}(C-1)+\frac{1}{6}(C-1)^2\right] = \pi a^2\left[1+C-1+\frac{1}{3}(C-1)^2\right]$$

$$= \pi a^2\left[C+\frac{1}{3}(C-1)^2\right]$$

So

$$\epsilon_t = \frac{1}{\pi a^2}\,\frac{(\pi a^2)^2\left(\frac{1}{2}+\frac{C}{2}\right)^2}{\pi a^2\left[C+\frac{1}{3}(C-1)^2\right]} = \frac{1}{4}\,\underline{\frac{(1+C)^2}{C+\frac{1}{3}(C-1)^2}} \qquad \text{QED}$$

(b) $n=2$

$$\mathcal{J}_1 = \left\{2\pi\int_0^a\left[C+(1-C)\left(1-\frac{1}{a^2}r^2\right)^2\right]r\,dr\right\}^2$$

$$= \left\{2\pi\int_0^a\left[Cr+(1-C)\left(r-\frac{2}{a^2}r^3+\frac{1}{a^4}r^5\right)\right]dr\right\}^2$$

$$= \left\{2\pi\left[C\frac{a^2}{2}+(1-C)\left(\frac{a^2}{2}-\frac{2}{a^2}\frac{a^4}{4}+\frac{1}{a^4}\frac{a^6}{6}\right)\right]\right\}^2 = \left\{\pi a^2\left[C+(1-C)\frac{1}{3}\right]\right\}^2$$

$$\mathcal{J}_2 = 2\pi\int_0^a\left\{C+(1-C)\left[1-\frac{r^2}{a^2}\right]^2\right\}^2 r\,dr$$

$$= 2\pi\int_0^a\left\{C^2 r + 2C(1-C)\left[r-\frac{2}{a^2}r^3+\frac{1}{a^4}r^5\right] + (1-C)^2\left[r-\frac{4}{a^2}r^3+\frac{6}{a^4}r^5-\frac{4}{a^6}r^7+\frac{1}{a^8}r^9\right]\right\}dr$$

after expanding

$$= 2\pi\left\{C^2\frac{a^2}{2}+2C(1-C)\left[\frac{a^2}{2}-\frac{2}{a^2}\frac{a^4}{4}+\frac{1}{a^4}\frac{a^6}{6}\right] + (1-C)^2\left[\frac{a^2}{2}-\frac{4}{a^2}\frac{a^4}{4}+\frac{6}{a^4}\frac{a^6}{6}-\frac{4}{a^6}\frac{a^8}{8}+\frac{1}{a^8}\frac{a^{10}}{10}\right]\right\}$$

$$= \pi a^2\left\{C^2+2C(1-C)\frac{1}{3}+(1-C)^2\frac{1}{5}\right\}$$

$$\epsilon_t = \frac{1}{\pi a^2}\,\frac{(\pi a^2)^2\left[C+\frac{1}{3}(1-C)\right]^2}{\pi a^2\left[C^2+\frac{2}{3}C(1-C)+\frac{1}{5}(1-C)^2\right]} = \frac{1}{9}\,\underline{\frac{(1+2C)^2}{C^2+\frac{2}{3}C(1-C)+\frac{1}{5}(1-C)^2}}$$

8.5-4 (con't)

(c) For a 10 dB edge taper $C=0.3162$

n=1

$$\epsilon_t = \frac{1}{4}\frac{(1+0.3162)^2}{0.3162+\frac{1}{3}(1-0.3162)^2} = \underline{0.91746}$$

n=2

$$\epsilon_t = \frac{1}{9}\frac{(1+2(0.3162))^2}{(0.3162)^2+\frac{2}{3}0.3162(1-0.3162)+\frac{1}{5}(1-0.3162)^2} = \underline{0.8769}$$

which both agree with Table 8-1 b.

8.6-1

From (8-147)

$$\rho(1+\cos\theta') = 2f$$

Squaring

$$\rho^2(1+2\cos\theta'+\cos^2\theta') = 4f^2$$

Squaring (8-148)

$$r'^2 = \rho^2\sin^2\theta' = \rho^2 - \rho^2\cos^2\theta' \quad \text{or} \quad \rho^2\cos^2\theta' = \rho^2 - r'^2$$

Then

$$\rho^2+2\rho^2\cos\theta'+\rho^2-r'^2 = 4f^2$$

$$2\rho\rho(1+\cos\theta')-r'^2 = 4f^2$$

Using (8-147) again

$$2\rho 2f - r'^2 = 4f^2$$

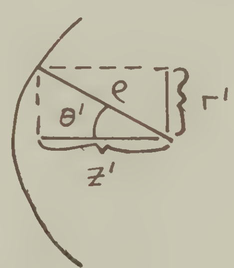
Thus

$$\rho = \frac{4f^2+r'^2}{4f} \quad \text{which is (8-158)}$$

8.6-2

From (8-147)

$$\rho = \frac{2f}{1+\cos\theta'}$$



Converting to coordinates for plotting

$$z' = \rho\cos\theta' \qquad r' = \rho\sin\theta'$$

So

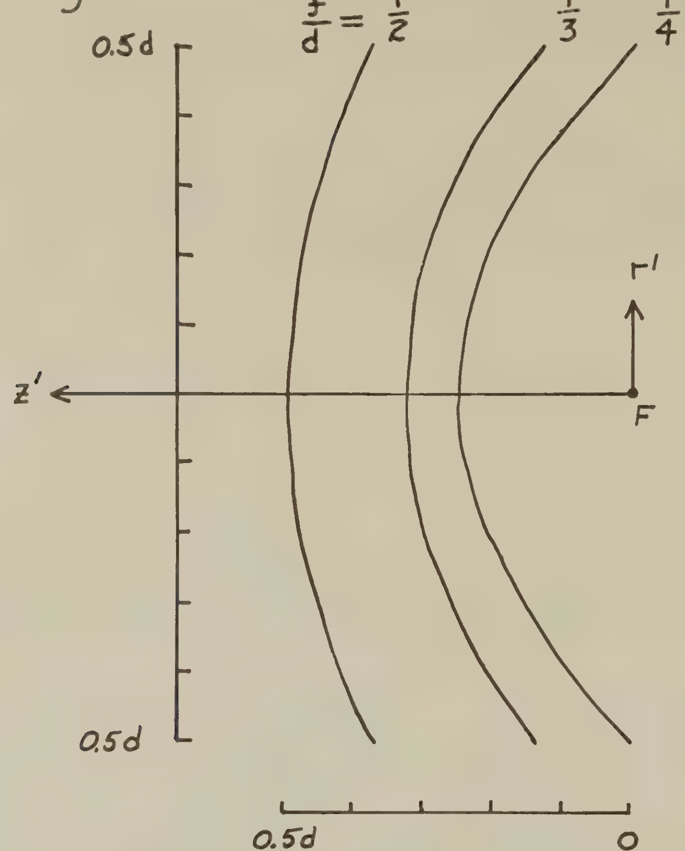
$$z' = 2f\frac{\cos\theta'}{1+\cos\theta'}, \qquad r' = 2f\frac{\sin\theta'}{1+\cos\theta'}$$

And

$$\frac{z'}{d} = 2\frac{f}{d}\frac{\cos\theta'}{1+\cos\theta'} \qquad \frac{r'}{d} = 2\frac{f}{d}\frac{\sin\theta'}{1+\cos\theta'}$$

8.6-2 (con't)

Evaluating these for θ from 0 to θ_o (where $\frac{r'}{d} = \frac{1}{2}$) and plotting



8.6-3

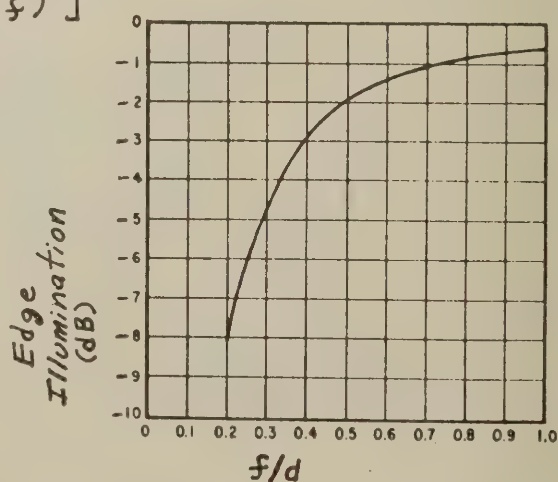
From (8-158)

$$\frac{1}{\rho} = \frac{1}{\frac{4f^2+r'^2}{4f}} = \left[f + f\left(\frac{r'}{2f}\right)^2\right]^{-1} = \frac{1}{f}\left[1+\left(\frac{r'}{2f}\right)^2\right]^{-1} \rightarrow \underset{\text{normalized}}{\left[1+\left(\frac{r'}{2f}\right)^2\right]^{-1}}$$

At the edge $r' = a = d/2$ and

$$\text{edge illumination} = \left[1+\frac{1}{16}\left(\frac{d}{f}\right)^2\right]^{-1}$$

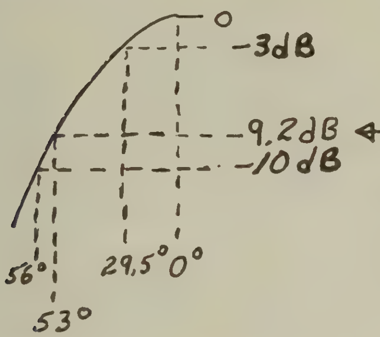
f/d	$\left[1+\frac{1}{16}\left(\frac{d}{f}\right)^2\right]^{-1}$	
0	0	$-\infty$ dB
0.1	0.138	−17.2 dB
0.2	0.390	−8.2
0.3	0.590	−4.9
0.4	0.719	−2.9
0.5	0.8	−1.9
0.6	0.852	−1.4
0.7	0.887	−1.04
0.8	0.911	−0.81
0.9	0.928	−0.65
1.0	0.941	−0.53



8.6-4 *H-plane of Example 8-8 antenna*

From (8-169) $\theta_0 = 53.1°$

From Table 8-2 in H-plane $\theta_{3dB} = \frac{59°}{2} = 29.5°$ $\theta_{10dB} = \frac{112°}{2} = 56°$



Linearly interpolating

$$\frac{56° - 29.5°}{7\,dB} = 3.786 \frac{deg}{dB}$$

$$0.264 \frac{dB}{deg} \rightarrow 0.79 \frac{dB}{3°} \simeq 0.8 \frac{dB}{3°}$$

$$-10 + (0.8) = -9.2\,dB$$

From (8-170) the free space loss at the reflector rim is

$$20 \log\left[1 + \tfrac{1}{16}(2)^2\right] = 1.9$$

So the edge illumination is

$$-9.2 - 1.9 = -11.1\ dB$$

Interpolating from Table 8-1b

$$HP = 1.1865 \frac{\lambda}{2a} = 1.1865 \frac{0.0105}{1.22} \frac{180°}{\pi} = 0.585°$$

which compares to 0.556° measured.

8.6-5

$\lambda = 0.1429\ m$ @ 2.1 GHz $d = 6\ ft = 1.829\ m$

From (8-84)

$$G = \epsilon_{ap} \frac{4\pi}{\lambda^2} A_p = \epsilon_{ap}\left(\frac{\pi d}{\lambda}\right)^2 = 0.55\left(\frac{\pi d}{\lambda}\right)^2 = 0.55\left(\frac{\pi\, 1.829}{0.1429}\right)^2$$

$$= 889.25 = \boxed{29.49\ dB}$$

(Andrews quotes 29.5 dB)

8.6-6

$\lambda = 0.02678\ m$ @ 11.2 GHz $d = 12\ ft = 3.658\ m$

$$G = 0.55\left(\frac{\pi d}{\lambda}\right)^2 = 0.55\left(\frac{\pi\, 3.658}{0.02678}\right)^2 = 101{,}281.2$$

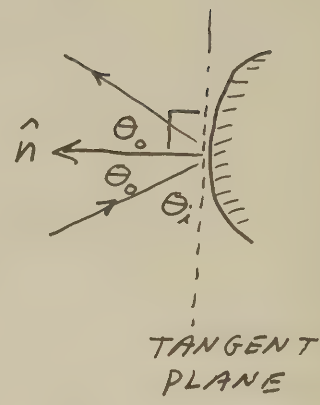
$$= \boxed{50.05\ dB}$$

(S-A model UDA-12-107 quotes 49.8 dB)

CHAPTER 9

9.1-1

$$\frac{1}{\rho^r_{1,2}} = \frac{1}{2}\left(\frac{1}{\rho^i_1} + \frac{1}{\rho^i_2}\right) + \frac{1}{f_{1,2}}$$



$\rho^i_1 = \rho^i_2$ (PLANE WAVE INCIDENCE)

$\theta_0 + \theta_i = 90°$

a) $\theta_1 = \theta_0$, $\theta_2 = 90°$

$$\frac{1}{\rho^r_{1,2}} = \frac{1}{2}\left(\frac{2}{\rho^i_1}\right) + \frac{1}{f_{1,2}}$$

$$\frac{1}{\rho^r_{1,2}} = \frac{1}{l_0} + \frac{1}{\cos\theta_0}\left[\frac{1}{r^c_1} + \frac{\cos^2\theta_0}{r^c_2}\right] \pm \sqrt{\frac{1}{\cos^2\theta_0}\left[\frac{1}{r^c_1} + \frac{\cos^2\theta_0}{r^c_2}\right]^2 - \frac{4}{r^c_1 r^c_2}}$$

WHEN $r^c_2 \to \infty$

$$\frac{1}{\rho^r_1} = \frac{1}{l_0} + \frac{2}{r^c_1 \cos\theta_0}, \qquad \frac{1}{\rho^r_2} = \frac{1}{l_0}$$

WHEN $r^c_1 \to \infty$

$$\frac{1}{\rho^r_{1,2}} = \frac{1}{l_0} + \frac{1}{\cos\theta_0}\left[\frac{\cos^2\theta_0}{r^c_2}\right] \pm \sqrt{\frac{1}{\cos^2\theta_0}\left[\frac{\cos^2\theta_0}{r^c_2}\right]^2}$$

$$\frac{1}{\rho^r_1} = \frac{1}{l_0} + \frac{2\cos\theta_0}{r^c_2}, \qquad \frac{1}{\rho^r_2} = \frac{1}{l_0}$$

b) $\rho^i_1 = \infty = \rho^i_2$

$$\therefore \frac{1}{\rho_1} = \frac{1}{f_1}, \qquad \frac{1}{\rho_2} = \frac{1}{f_2}$$

$$\frac{1}{\rho_1\rho_2} = \frac{1}{f_1}\frac{1}{f_2} = \frac{4}{r^c_1 r^c_2}$$

$$\therefore \sqrt{\rho_1\rho_2} = \frac{1}{2}\sqrt{r^c_1 r^c_2}$$

9.1-2

SEE Fig. 9-3

$$S_0\, d\sigma_0 = S\, d\sigma$$

IF $d\sigma_0$ IS A LARGE DISTANCE ℓ FROM $d\sigma$, THEN WE CAN LET r BE THE APPROXIMATE DISTANCE FROM $d\sigma$ TO BOTH CAUSTICS.

$$\therefore\ d\sigma = \frac{r}{\rho_1}\frac{r}{\rho_2}\, d\sigma_0$$

$$\frac{S}{S_0} = \frac{d\sigma_0}{d\sigma} = \frac{d\sigma_0}{\frac{r^2}{\rho_1\rho_2}\, d\sigma_0} = \frac{\rho_1\rho_2}{r^2} \qquad \text{BUT } \sqrt{\rho_1\rho_2} = \tfrac{1}{2}\sqrt{r_1^c r_2^c}$$

$$\frac{S}{S_0} = \frac{r_1^c r_2^c}{4r^2}$$

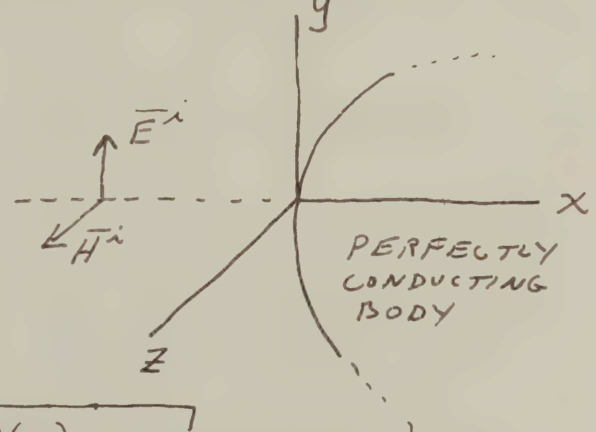
$$\sigma = 4\pi r^2 \frac{S}{S_0} = 4\pi r^2 \frac{r_1^c r_2^c}{4r^2} = \pi r_1^c r_2^c$$

9.1-3

$$\frac{1}{\rho_1} = \frac{2}{r_c}$$

$$\frac{1}{\rho_1} = \frac{2}{5\lambda}, \quad \rho_1 = 2.5\lambda$$

$$\frac{1}{\rho_2} = \frac{2}{10\lambda}, \quad \rho_2 = 5\lambda$$



$$\therefore\ \bar{E}^s = -\hat{y}\, e^{+j\beta x}\sqrt{\frac{(2.5)(5)}{(2.5+\ell)(5+\ell)}}$$

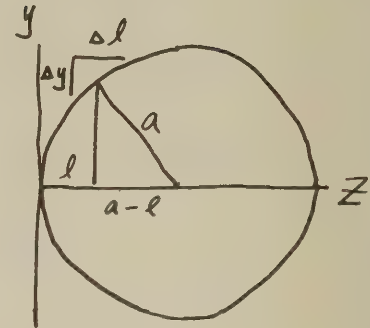
$$\bar{H}^s = \hat{z}\,\frac{e^{+j\beta x}}{\eta}\sqrt{\frac{12.5}{(2.5+\ell)(5+\ell)}}$$

9.2-1

$$\sigma = \frac{4\pi}{\lambda^2}\left|\frac{1}{H^i}\iint(\hat{n}\times\bar{H}^i)\,e^{-j\beta\ell}\,ds\right|^2 \qquad (9\text{-}30)$$

$$\sigma = \frac{4\pi}{\lambda^2}\left|\iint ds\right|^2 = \frac{4\pi A^2}{\lambda^2}$$

9.2-2



$$\frac{4\pi}{\lambda^2}\left|\int_0^a e^{-j2\beta\ell}\,\frac{dS_z}{d\ell}\,d\ell\right|^2 \overset{?}{=} \pi a^2$$

$$(\ell-a)^2+y^2=a^2$$

$$y^2 = a^2-(\ell-a)^2$$

$$2y\frac{dy}{d\ell} = -2(\ell-a)$$

$$2\pi y\frac{dy}{d\ell} = -2(\ell-a)\pi = \frac{dS_z}{d\ell}$$

$$\sigma = \frac{4\pi}{\lambda^2}\left|2\pi\int_0^a e^{-j2\beta\ell}(a-\ell)\,d\ell\right|^2 \qquad \sim(9\text{-}33)$$

$$\sigma = \pi a^2\left|\frac{1}{j}\left(1-\frac{1}{2j\beta a}\right)\right|^2 \xrightarrow[\beta a\to\infty]{} \pi a^2$$

9.2-3

$$\sigma = \frac{4\pi}{\lambda^2}\left|\int_0 e^{-j2\beta\ell}\,\frac{dS_z}{d\ell}\,d\ell\right|^2$$

$$dS_z = \ell\,\mathrm{TAN}\,\alpha\cdot 2\pi\cdot d\ell\,\mathrm{TAN}\,\alpha \qquad \mathrm{TAN}\,\alpha = \frac{y}{\ell}$$

$$\frac{dS_z}{d\ell} = 2\pi\ell\,\mathrm{TAN}^2\alpha$$

$$\sigma = \frac{4\pi}{\lambda^2}\left|\int_0 \lambda\, e^{-j2\beta\ell}\left(\frac{2\pi}{\lambda}\ell\right)\mathrm{TAN}^2\alpha\,d\ell\right|^2$$

$$\sigma = \frac{4\pi}{\lambda^2}\left|j\lambda\,\frac{\mathrm{TAN}^2\alpha}{2}\left[e^{-j2\beta\ell}\ell - \frac{e^{-j2\beta\ell}}{-j2\beta}\right]_0^\infty\right|^2 = \frac{\lambda^2\,\mathrm{TAN}^4\alpha}{16\pi}$$

9.2-3 (CONT)

AN ALTERNATIVE FORM TO THE EXPRESSION IN PROB 9.2-2 IS

$$\sigma = \frac{4\pi}{\lambda^2}\left|\int_0^{L'}\int_0^{2\pi} e^{-j2\beta l}\, \rho\frac{\partial\rho}{\partial l}\, d\phi\, dl\right|^2$$

$$\tan\alpha = \frac{\rho}{l}, \quad \therefore \quad \rho = l\tan\alpha, \quad d\rho = dl\tan\alpha, \quad \frac{d\rho}{dl} = \tan\alpha$$

$$\sigma = \frac{4\pi}{\lambda^2}\left|\int_0^{\infty}\int_0^{2\pi} e^{-j2\beta l}\tan\alpha\, d\phi\, \frac{\rho\, d\rho}{\tan\alpha}\right|^2$$

$$\sigma = \frac{4\pi}{\lambda^2}\left|2\pi\int_0^{\infty} e^{-j2\beta\rho\cot\alpha}\, \rho\, d\rho\right|^2$$

$$\int_0^{\infty} x e^{-ax}\, dx = \frac{1}{a^2}$$

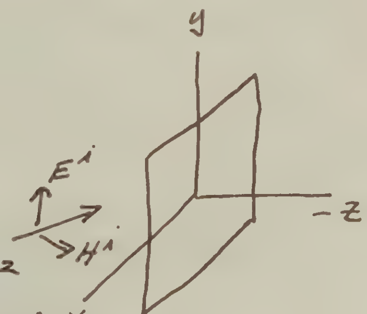
$$\int x e^{ax}\, dx = e^{ax}\left[\frac{x}{a} - \frac{1}{a^2}\right]$$

$$\sigma = \frac{4\pi}{\lambda^2}\left|2\pi\, \frac{1}{(j2\beta\cot\alpha)^2}\right|^2$$

$$\sigma = \frac{4\pi}{\lambda^2}\left|\frac{\lambda^2\tan^2\alpha}{-4(2\pi)}\right|^2 = \frac{\lambda^2\tan^4\alpha}{16\pi}$$

9.2-4

$$\sigma = \frac{4\pi}{\lambda^2}\frac{1}{|H^i|^2}\left|\iint(\hat{n}\times\bar{H}^i)\, e^{-j\beta l}\, ds\right|^2$$



$$\hat{n} = \hat{z}, \quad \bar{H}^i = \hat{x}\cos\theta - \hat{z}\sin\theta$$

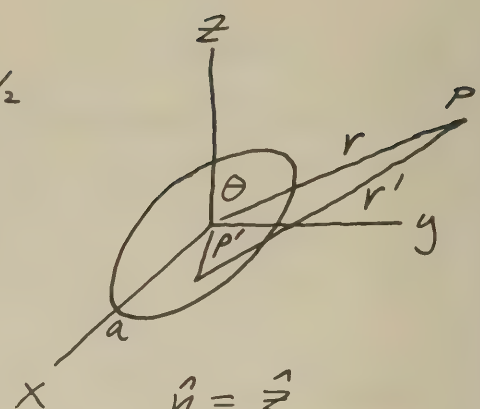
$$\sigma = \frac{4\pi}{\lambda^2}\left|\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} e^{-j2\beta x'\sin\theta}\, dx'\, dy'\right|^2\cos^2\theta$$

$$\sigma = \frac{4\pi}{\lambda^2}\, a^2 a^2\left[\frac{\sin(\beta a\sin\theta)}{\beta a\sin\theta}\right]^2\cos^2\theta$$

$$\sigma = \frac{4\pi a^4}{\lambda^2}\left[\frac{\sin(\beta a\sin\theta)}{\beta a\sin\theta}\right]^2\cos^2\theta$$

9.2-5

$$r' = \left[r^2 + \rho'^2 - 2r\rho' \sin\theta \cos(\phi - \phi')\right]^{1/2}$$



$$r' \cong r - \rho' \sin\theta \cos(\phi - \phi') + \frac{\rho'^2}{2r}\left[1 - \sin^2\theta \cos^2(\phi - \phi')\right]$$

FOR THE FAR FIELD

$$r' \approx r - \rho' \sin\theta \cos(\phi - \phi')$$

$$\hat{n} = \hat{z}$$

$$\sigma = \frac{4\pi}{\lambda^2}\left| \cos\theta \iint e^{-j2\beta\rho' \sin\theta \cos(\phi - \phi')} \rho' d\phi' d\rho' \right|^2 \qquad \bar{H}^i = \hat{x}\cos\theta - \hat{z}\sin\theta$$

$$\int_0^{2\pi} e^{2j\beta\rho' \sin\theta \cos(\phi - \phi')} d\phi' = 2\pi J_0(2\beta\rho' \sin\theta)$$

$$2\pi \int_0^a J_0(2\beta\rho' \sin\theta)\, \rho' d\rho' = \frac{2\pi a^2 J_1(2\beta a \sin\theta)}{2\beta a \sin\theta}$$

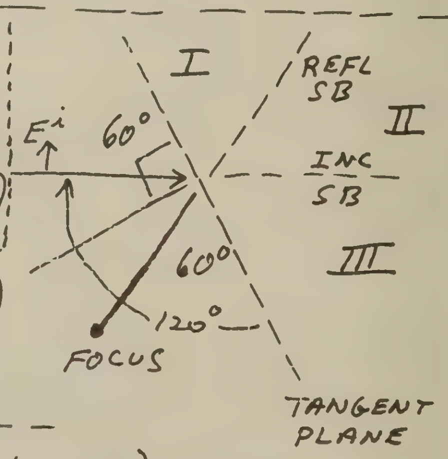
$$\sigma = \frac{4\pi}{\lambda^2} \cos^2\theta \left| \frac{\pi a^2 J_1(2\beta a \sin\theta)}{\beta a \sin\theta} \right|^2 \qquad \boxed{J_1(x) = \frac{x}{2} \text{ AS } x \to 0}$$

$$\sigma = \pi \cos^2\theta \left| \frac{a\, J_1(2\beta a \sin\theta)}{\sin\theta} \right|^2 = \frac{\pi a^2}{\tan^2\theta}\left[J_1\left(\frac{4\pi a \sin\theta}{\lambda}\right)\right]$$

9.3-1

$$E(\rho, \phi) = + \mathcal{N}_*^r(\rho, \phi + 120°) + \mathcal{N}_B^r(\rho, \phi + 120°) + \mathcal{N}_*^i(\rho, \phi - 120°) + \mathcal{N}_B^i(\rho, \phi - 120°)$$

REGION I



$$E(\rho, \phi) = \mathcal{N}_B^r(\rho, \phi + 120°) + \mathcal{N}_*^i(\rho, \phi - 120°) + \mathcal{N}_B^i(\rho, \phi - 120°)$$

REGION II

$$E(\rho, \phi) = \mathcal{N}_B^r(\rho, \phi + 120°) + \mathcal{N}_B^i(\rho, \phi - 120°) \text{ and}$$

$$E(\rho, \phi) \approx \mathcal{N}_B^i(\rho, \phi - 120°)$$ BECAUSE $\mathcal{N}_B^r$ WILL BE NUMERICALLY SMALL

REGION III

9.3-2

a) $$\int_0^\infty e^{-j\tau^2}d\tau = \int_0^\infty \cos\tau^2 d\tau - j\int_0^\infty \sin\tau^2 d\tau$$

$$= \frac{\pi}{2\sqrt{2}} - j\frac{\pi}{2\sqrt{2}} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,e^{-j\frac{\pi}{4}}$$

b) $$\int_0^x e^{-j\tau^2}d\tau = \sqrt{\frac{\pi}{2}}\,C_1(x) - j\sqrt{\frac{\pi}{2}}\,S_1(x)$$

$$= \sqrt{\frac{\pi}{2}}\,C\left(x\sqrt{\frac{2}{\pi}}\right) - j\sqrt{\frac{\pi}{2}}\,S\left(x\sqrt{\frac{2}{\pi}}\right)$$

$$\int_0^5 e^{-j\tau^2}d\tau = \sqrt{\frac{\pi}{2}}\,C\left(5\cdot\sqrt{\frac{2}{\pi}}\right) - j\sqrt{\frac{\pi}{2}}\,S\left(5\cdot\sqrt{\frac{2}{\pi}}\right)$$

$$= \sqrt{\frac{\pi}{2}}\,(0.5636312) - j\sqrt{\frac{\pi}{2}}\,(0.4991914)$$

c) $$\int_5^\infty e^{-j\tau^2}d\tau = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,e^{-j\frac{\pi}{4}} - \int_0^5 e^{-j\tau^2}d\tau$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}}\,e^{-j\frac{\pi}{4}} - \left\{\sqrt{\frac{\pi}{2}}\,(0.5636312 - j0.4991914)\right\}$$

9.3-3

$$V_B(\rho,\phi^\pm) = \frac{e^{-j(\beta\rho+\pi/4)}}{\sqrt{2\pi\beta\rho}}\;\frac{(1/n)\sin(\pi/n)}{\cos(\pi/n)-\cos(\phi^\pm/n)} \qquad (9-52)$$

$$V_B(\rho,\phi^\pm) = \frac{2e^{j(\pi/4)}}{n\sqrt{\pi}}\;\frac{\sin(\pi/n)}{\cos(\pi/n)-\cos(\phi^\pm/n)}\left|\cos\frac{\phi^\pm}{2}\right|$$

$$\cdot\, e^{j\beta\rho\cos\phi^\pm}\int_{\sqrt{\alpha\beta\rho}}^\infty e^{-j\tau^2}d\tau + \cdots \qquad (9-54)$$

a) $\phi^+ = \phi + \phi' = 220° + 45° = 265° = 1.472\pi$

$\phi^- = \phi - \phi' = 220° - 45° = 175° = 0.972\pi$

$\rho = 10\lambda$

9.3-3 (CONT.)

$$N_B(\rho,\phi^+) = \frac{e^{-j(20\pi+\pi/4)}}{2\pi\sqrt{10}\,(1.5)} \; \frac{\sin\left(\frac{\pi}{1.5}\right)}{\cos\left(\frac{\pi}{1.5}\right)-\cos\left(\frac{1.472\pi}{1.5}\right)}$$

$$N_B(\rho,\phi^+) = +0.05832 \angle -45^\circ \qquad (9\text{-}52)$$

$$N_B(\rho,\phi^-) = -0.5672 \angle -45^\circ \qquad (9\text{-}52)$$

$$N_B(\rho,\phi^+) = \frac{4}{3\sqrt{\pi}} \; \frac{0.866}{0.498} \, |0.676| \; 0.066 \, e^{-j45^\circ} e^{-j20\pi}$$

$$= -0.058 \angle -45^\circ \qquad (9\text{-}54)$$

$$N_B(\rho,\phi^-) = e^{j\pi/4} \; \frac{0.752\times 0.866}{-0.0511} \; 0.44 \, e^{j0.08\pi} \; 0.413 \, e^{j84^\circ}$$

$$= 0.232 \angle -36.6^\circ \qquad (9\text{-}54)$$

b)

$$\phi^+ = 230^\circ + 45^\circ = 275^\circ = 1.527\pi$$

$$\phi^- = 230^\circ - 45^\circ = 185^\circ = 1.027\pi$$

$$N_B(\rho,\phi^+) = \frac{1}{19.87} \; \frac{0.577}{-0.5+0.998} \; e^{-j\pi/4}$$

$$= 0.058 \angle -45^\circ \qquad (9\text{-}52)$$

$$N_B(\rho,\phi^-) = 0.577 \angle -45^\circ \qquad (9\text{-}52)$$

$$N_B(\rho,\phi^+) = \frac{0.752\times 0.866}{-0.5+0.998} \; 0.737\times 0.061 \, e^{-j\beta\rho(1+\cos\phi^+)} \cdot e^{j\beta\rho\cos\phi^+} e^{-j\pi/2} e^{j\pi/4}$$

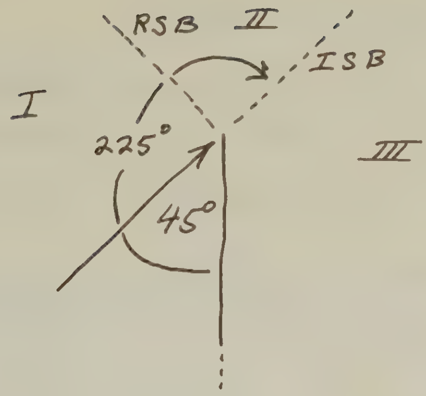
$$= 0.058 \angle -45^\circ \qquad (9\text{-}54)$$

$$N_B(\rho,\phi^-) = \frac{0.752\times 0.866\times 0.046}{0.04952} \; e^{j\pi/4} e^{-j19.42\pi} e^{j84^\circ} \; 0.413$$

$$= 0.239 \angle 143.4^\circ$$

(9.3-3) (CONT)

THE RESULTS USING (9-52) DIFFER FROM THOSE USING (9-54) BECAUSE THE OBSERVATION POINT IS NEAR THE INCIDENT FIELD SHADOW BOUNDARY (ISB).



$N_*^r = 0$ IN (a) , $N_*^i = 0 = N_*^r$ IN (b).

9.3-4

a) $\phi = 90°$ $\quad \phi + \phi' = 135°$
$\phi' = 45°$ $\quad \phi - \phi' = 45°$
$n = 1.5$

$$\frac{2\pi}{\lambda} 10\lambda \left(\cos\frac{180}{1.5} - \frac{\cos(135° \text{ OR } 45°)}{1.5} \right) >> 1$$

SO (9-52) MAY BE USED.

$$N_B(\rho, \phi^\pm) = \frac{e^{-j(20\pi + \pi/4)}}{\sqrt{2\pi \cdot \frac{2\pi}{\lambda} \cdot 10\lambda}} \quad \frac{\frac{1}{1.5}\sin(120°)}{\cos(120°) - \cos(90° \text{ OR } 30°)}$$

$$N_B(10, \phi^+) = -0.058\, e^{-j\pi/4}$$

$$N_B(10, \phi^-) = -0.021\, e^{-j\pi/4}$$

c) $\phi = 180°$ $\quad \phi + \phi' = 225°$ | OBS. PT. IS NOT NEAR A SB.
$\phi' = 45°$ $\quad \phi - \phi' = 135°$

$$N_B(\rho, \phi^\pm) = \frac{e^{-j(20\pi + \pi/4)}}{2\pi\sqrt{10}} \quad \frac{\frac{1}{1.5}\sin\left(\frac{180}{1.5}\right)}{\cos\left(\frac{180}{1.5}\right) - \cos\left(\frac{225° \text{ OR } 135°}{1.5}\right)}$$

$$N_B(\rho, \phi^+) = 0.0383\, e^{-j\pi/4}$$

$$N_B(\rho, \phi^-) = -0.024\, e^{-j\pi/4}$$

$$\left[\therefore E^d(\rho, \phi) = \pm\, 0.0383\, e^{-j\pi/4} - 0.024\, e^{-j\pi/4}\right]$$

9.3-4 (CONT)

$\phi = 138^\circ$ $\quad\quad \phi + \phi' = 138^\circ + 45^\circ = 183^\circ$

$\phi' = 45^\circ$ $\quad\quad \phi - \phi' = 138^\circ - 45^\circ = 93^\circ$

USING (9-54) BECAUSE WE ARE NEAR A SB,

$$N_B(\rho, \phi^\pm) = \frac{2e^{j\pi/4}\, 0.866}{1.5\sqrt{\pi}\begin{pmatrix}-0.5+0.53\\ -0.5-0.47\end{pmatrix}} \times \begin{pmatrix}-0.026\\ +0.688\end{pmatrix} \cdot e^{-j20\pi\begin{pmatrix}-0.998\\ -0.052\end{pmatrix}} \int_{\sqrt{\alpha^\pm 20\pi}}^{\infty} e^{-j\tau^2}\, d\tau$$

$$\sqrt{\alpha^\pm 20\pi} = \begin{matrix}0.293\\ 8.13\end{matrix}$$

$$0.293\sqrt{\frac{2}{\pi}} = 0.23378$$

$$8.13\sqrt{\frac{2}{\pi}} = 6.4868$$

$$\int_{\sqrt{\alpha^\pm 20\pi}}^{\infty} e^{-j\tau^2}\, d\tau = \frac{1}{2}\sqrt{\frac{\pi}{2}}\, e^{-j\pi/4} - \int_0^{\sqrt{\alpha^\pm 20\pi}} e^{-j\tau^2}\, d\tau$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}}\, e^{-j\pi/4} - \sqrt{\frac{\pi}{2}}\begin{pmatrix}C(0.23378) - jS(0.23378)\\ C(6.4868) - jS(6.4868)\end{pmatrix}$$

$$= \begin{matrix}1.549 - j0.4422\\ -0.176 - j1.131\end{matrix} = \begin{matrix}0.4685\, e^{-j0.393\pi}\\ 0.4668\, e^{j1.049\pi}\end{matrix}$$

$$\therefore N_B(\rho, \phi^\pm) = \begin{matrix}0.265\, e^{-j0.116\pi}\\ 0.22\, e^{j0.896\pi}\end{matrix}$$

$$E^d(\rho, \phi) = \pm N_B^{\,r}(\rho, \phi^+) + N_B^{\,i}(\rho, \phi^-)$$

9.3-5

$$S_i = \frac{P_t G_t}{4\pi r_1^2} = \frac{10\times10^3\times10}{4\pi(10.05\times10^3)^2} = 7.879\times10^{-5}$$

$$E^i = \sqrt{\eta S_i} = 0.1723 \text{ V/m}$$

$$E_\perp^d = -E^i D_\perp \sqrt{\frac{\rho}{s(\rho+s)}}\, e^{-j\beta s} = -0.1723\sqrt{\frac{10.05}{3.162(10.05\times3.162)}}\, e^{-j\frac{2\pi}{0.5}\times3.162\times10^3} \times\left(-0.0956\, e^{-j\pi/4}\right)$$

$$E_\perp^d = 8.078\times10^{-3} e^{-j\pi/4} \text{ V/m}$$

$$S_r = \frac{|E_\perp^d|^2}{\eta} = 0.173\times10^{-6} \text{ W/m}^2$$

$$P_r = A_e S_r = \frac{G_r\lambda^2}{4\pi} S_r = 0.0275\times10^{-6} \text{ W}$$

9.3-6

$$E(\rho,\phi) = \pm\left[N_*^r(\rho,\phi+\phi') + N_B^r(\rho,\phi+\phi')\right] + N_*^i(\rho,\phi-\phi') + N_B^i(\rho,\phi-\phi')$$

= g.o. REFLECTED FIELD + (REFLECTED) DIFFRACTED FIELD + g.o. INCIDENT FIELD + (INCIDENT) DIFFRACTED FIELD

9.4-1

FROM (9-5f), SINCE E IS $\perp$ EDGE,

$$E_\perp^d(\rho) = -D_\perp E_{\perp'}^i(Q)\, A(s)\, e^{-j\beta s}, \quad s=\rho$$

$$E_\perp^d(\rho) = -D_\perp E_{\perp'}^i(Q)\, A(\rho)\, e^{-j\beta\rho}$$

FROM FIG. 9-3 $|E| = |E_0|\sqrt{\frac{\rho_1}{\rho_1+\rho}} \xrightarrow[\rho\to\infty]{} \frac{1}{\sqrt{\rho}} = A(\rho)$

ALTERNATIVELY, THE DIFFRACTED FIELD APPEARS TO ORIGINATE FROM A LINE SOURCE, SO $|E^d| \propto \frac{1}{\sqrt{\rho}}$

$$\therefore\ E_\perp^d(\rho) = -D_\perp E_{\perp'}(Q)\frac{1}{\sqrt{\rho}}\, e^{-j\beta\rho}$$

9.4-2

FOR A MAGNETIC LINE SOURCE E IS ∥ EDGE.

∴ FROM (9-58)

$$E_{\parallel}^{d}(P) = -D_{\parallel}\,E_{\parallel}^{i}(Q)\;A(s)\,e^{-j\beta s} \quad , \quad A(s) = \frac{1}{\sqrt{\rho}}$$

$$\therefore\; E_{\parallel}^{d}(P) = -D_{\parallel}\,E_{\parallel}^{i}(Q)\,\frac{1}{\sqrt{\rho}}\,e^{-j\beta\rho}$$

9.4-3

SPREADING IN THE PLANE CONTAINING THE EDGE IS $\sqrt{\frac{s'}{s'+s}}$. SPREADING IN THE ORTHOGONAL PLANE IS $\frac{1}{\sqrt{s}}$ AND RESULTS FOLLOW.

9.4-4

A GENERAL "EDGE FIXED" CO-ORDINATE SYSTEM IS DEFINED BY A SET OF VECTORS $\hat{T}, \hat{N}, \hat{B}$, WHERE $\hat{T}$ IS A UNIT VECTOR PARALLEL TO THE EDGE, $\hat{N}$ IS A UNIT VECTOR NORMAL TO THE EDGE AND $\hat{B}$ IS THE UNIT VECTOR BINORMAL TO THE EDGE AND POINTING INTO THE SHADOWED HALF-SPACE.

$$\therefore\quad \hat{T} = -\hat{z}\;,\quad \hat{N} = -\hat{x}\cos\delta + \hat{y}\sin\delta\,,\quad \hat{B} = -\hat{x}\sin\delta - \hat{y}\cos\delta$$

LET (β,α) BE THE ANGLE OF INCIDENCE AND (β,θ) BE THE ANGLE OF DIFFRACTION. THEN THE DIFFRACTION MATRIX IS

$$\begin{bmatrix} -(X-Y) & 0 & 0 \\ (X-Y)\cot\beta\sin\theta & (X+Y)\cos\theta\cos\alpha & (X+Y)\cos\theta\sin\alpha \\ -(X-Y)\cot\beta\cos\theta & (X+Y)\sin\theta\cos\alpha & (X+Y)\sin\theta\sin\alpha \end{bmatrix}$$

$$X = \frac{1}{n}\sin\frac{\pi}{n}\left[\cos\frac{\pi}{n} - \cos\frac{1}{n}(\alpha-\theta)\right]^{-1}$$

$$Y = \frac{1}{n}\sin\frac{\pi}{n}\left[\cos\frac{\pi}{n} + \cos\frac{1}{n}(\pi+2\delta-\alpha-\theta)\right]^{-1}$$

AND $n = 2\left(1-\frac{\Omega}{\pi}\right)$, Ω = HALF THE INCLUDED ANGLE OF WEDGE

9.5-1

WHEN $X >> 0$, $F(X) = 2j|\sqrt{X}|e^{jX}\int_{|\sqrt{X}|}^{\infty} e^{-j\tau^2}d\tau \approx 1$

WE KNOW THAT

$$\cot(x+y) + \cot(x-y) = \frac{2\sin 2x}{\cos 2y - \cos 2x}$$

$$\therefore \frac{\sin(\pi/N)}{\cos(\pi/N) - \cos\left(\frac{\phi^{\mp}}{N}\right)} = -\frac{1}{2}\left[\frac{2\sin(\pi/N)}{\cos(\phi^{\mp}/N) - \cos(\pi/N)}\right]$$

WHERE $\phi^{\mp} = \phi \mp \phi'$

AND (9-69) FOLLOWS. WRITING (9-69) AS

$$D_{\substack{\parallel \\ \perp}} = \frac{e^{-j\pi/4}\sin(\pi/n)}{n\sqrt{2\pi\beta}\,\sin\gamma_0'}\left[\qquad\qquad\right]$$

$\gamma_0' = 90°$

$$E^d_{\substack{\parallel \\ \perp}} = -D_{\substack{\parallel \\ \perp}} E^i_{\substack{\parallel \\ \perp}} A(s)\, e^{-j\beta\rho}, \quad A(s) = \frac{1}{\sqrt{\rho}}$$

$$\therefore E^d_{\substack{\parallel \\ \perp}} = -\frac{e^{-j(\beta\rho + \pi/4)}}{\sqrt{2\pi\beta\rho}}\,\frac{1}{n}\sin(\pi/n)\left[\qquad\qquad\right]$$

AND $V_B(\rho, \phi^{\pm})$ FOLLOWS.

9.5-2 $\quad \beta_0' = 90°$, $L^i = L^r = \rho$ FOR PLANE WAVE INCIDENCE

$$a^{\pm}(\phi \pm \phi') = 2\cos^2\left(\frac{2n\pi N^{\pm} - (\phi \mp \phi')}{2}\right)$$

WHEN $n = 2$, $N^{\pm}$ = INTEGRAL VALUES

$$a^{\pm}(\phi \pm \phi') = 2\cos^2\left(\frac{\phi \pm \phi'}{2}\right) = 1 + \cos(\phi \pm \phi') = a(\phi \pm \phi')$$

SINCE $\cot(x+y) + \cot(x-y) = \dfrac{2\sin 2x}{\cos 2y - \cos 2x}$ THEN

$$\cot\left(\frac{\pi + \phi^{\mp}}{2n}\right) + \cot\left(\frac{\pi - \phi^{\mp}}{2n}\right) = \frac{\sin(\pi/N)}{\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\phi^{\mp}}{n}\right)}$$

9.5-2 (CONT.)

$$D_{\parallel,\perp}(\phi,\phi',\beta_0) = \frac{e^{-j\pi/4}\, 2\sin\left(\frac{\pi}{n}\right)}{2n\sqrt{2\pi\beta\rho}} \left\{ \frac{F[\beta\rho a(\phi-\phi')]}{\cos\frac{\pi}{n}-\cos\left(\frac{\phi-\phi'}{n}\right)} \mp \frac{F[\beta\rho a(\phi+\phi')]}{\cos\frac{\pi}{n}-\cos\left(\frac{\phi+\phi'}{n}\right)} \right\}$$

$$F(\beta\rho a) = 2j\sqrt{\beta\rho a}\; e^{j\beta\rho a}\int_{\sqrt{\beta\rho a}}^{\infty} e^{-j\tau^2}\, d\tau$$

$$D_{\parallel,\perp}(\phi,\phi',\beta_0) = \frac{2e^{j\pi/4}}{n\sqrt{\pi}} \left\{ \frac{\sin(\pi/n)}{\cos(\pi/n)-\cos\left(\frac{\phi-\phi'}{n}\right)} \left|\cos\frac{\phi^+}{2}\right| e^{j\beta\rho\cos\phi^+} \right.$$

$$\cdot \int_{\sqrt{a(\phi^+)\beta\rho}}^{\infty} e^{-j\tau^2}d\tau \pm \frac{\sin(\pi/n)}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi+\phi'}{n}\right)} \left|\cos\frac{\phi^-}{2}\right|$$

$$\left. \cdot\, e^{j\beta\rho\cos\phi^-} \int_{\sqrt{a(\phi^-)\beta\rho}}^{\infty} e^{-j\tau^2} d\tau \right\}$$

b) YES, BECAUSE THE OBS. PT. IS NOT IN THE CAUSTIC REGION.

c) NO, GTD OR UTD FAILS IN REGION OF CAUSTICS

9.5-3

$$2\pi n N^+ - (\phi \pm \phi') = \pi$$

$$\therefore\; N^+ = \frac{\pi + (\phi\pm\phi')}{2\pi n} = \frac{\pi}{2\pi n} + \frac{(\phi\pm\phi')}{2\pi n} = \frac{1}{2n} + \frac{(\phi\pm\phi')}{2\pi n}$$

AND N^+ TAKES ON VALUE OF NEAREST INTEGER. FOR EXAMPLE, IF $n=\frac{3}{2}$ (i.e. 90° WEDGE)

$$N^+ = \frac{1}{3} + \frac{\phi\pm\phi'}{3\pi} \implies$$ FOR $\phi\pm\phi' < 0.5\pi$, $N^+ = 0$

FOR $\frac{\pi}{2} \le \phi\pm\phi' < \frac{21\pi}{6}$, $N^+ = 1$

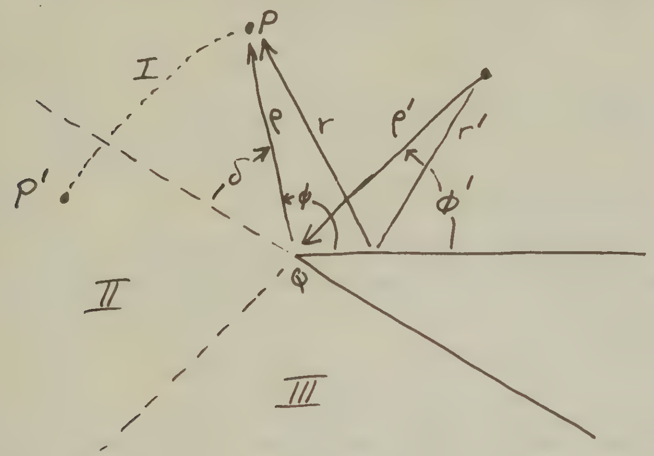
FOR N^-

$$N^- = \frac{-\pi + (\phi\pm\phi')}{2\pi n} = \frac{-\pi}{2\pi n} + \frac{(\phi\pm\phi')}{2\pi n} = \frac{-1}{2n} + \frac{(\phi\pm\phi')}{2\pi n}$$

NOTE! THE RELATIONSHIPS FOR N^+ AND N^- GIVEN IN THIS PROB. MAY BE EASIER FOR THE STUDENT TO USE.

9.5-4

a) WE WISH TO SHOW THAT THE DIFFRACTED FIELD MUST HAVE A DISCONTINUITY PROPORTIONAL TO $\dfrac{e^{-j\beta(\rho+\rho')}}{\sqrt{\rho+\rho'}}$



In the above diagram 0 is an electric or magnetic line source, illuminating a wedge whose edge Q is parallel to the line source. RB and SB are shadow boundaries which divide space into regions I illuminated by direct, reflected and diffracted rays, II by direct and diffracted rays and III by diffracted rays.

The reflected ray when it exists is reflected at R and its total length is r' + r. The incident (diffracted) ray has length $\rho'(\rho)$ and makes an angle $\phi'(\phi)$ to one of the wedge faces.

If 0 is an electric source we consider the $\bar{E}$ field
If 0 is a magnetic source we consider the $\bar{H}$ field

In each case the field is parallel to the edge and we are not concerned with its vector nature.

The total field at P is

$$u(P) = u^i\,(P) + u^r(P) + u^d\,(P) \tag{1}$$

where u^i is the direct field,

u^r is the field reflected from the wedge at R,and

u^d is the field diffracted from the edge at Q.

We note that u^i is continuous as P moves across RB to P'. u^r falls abruptly to zero at RB so that

$$\Delta u^r = -\,u^r\,(RBI) \tag{2}$$

where u^r (RBI) is the value of u^r on side I of RB, and

u^d must undergo a discontinuity

$$\Delta u^d = u^r\,(RBI) \tag{3}$$

The field radiated from 0 is a cylindrical wave which may be described by the formula

$$u^{i}(R) = A\frac{e^{-jkR}}{\sqrt{R}} \tag{4}$$

at a radial distance R from 0.

Then the reflected field at P is

$$u^{r}(P) = \pm A\frac{e^{-jk(r+r')}}{\sqrt{r+r'}}$$

$$\simeq \pm A\frac{e^{-jk(\rho+\rho')}}{\sqrt{\rho+\rho'}} \tag{5}$$

where P is close to RB. Here and subsequently the upper (lower) sign or other alternative corresponds to the electric (magnetic) line source. The approximation becomes exact when P reaches RB.

Thus

$$\Delta u^{r} = A\frac{e^{-jk(\rho+\rho')}}{\sqrt{\rho+\rho'}} \tag{6}$$

The diffracted field must suffer a discontinuity whose value is the negative of this. ——— Answer (a)

The diffracted field at P is

$$u^{d}(P) = u^{i}(Q)\begin{pmatrix} D_s \\ D_h \end{pmatrix}\frac{e^{-jk\rho}}{\sqrt{\rho}}$$

$$= A\frac{e^{-jk\rho'}}{\sqrt{\rho'}}\begin{pmatrix} D_s \\ D_h \end{pmatrix}\frac{e^{-jk\rho}}{\sqrt{\rho}}$$

$$= A\begin{pmatrix} D_s \\ D_h \end{pmatrix}\frac{e^{-jk(\rho+\rho')}}{\sqrt{\rho\rho'}} \tag{7}$$

NOTE: USE EITHER D_s ($D_{\parallel}$) OR D_h ($D_{\perp}$) DEPENDING ON WHETHER THE LINE SOURCE IS ELECTRIC OR MAGNETIC RESPECTIVELY

where

$$\begin{pmatrix} D_s \\ D_h \end{pmatrix} = \frac{-e^{-j\frac{\pi}{4}}}{2n\sqrt{2\pi k}}\left[\cot\left(\frac{\pi+(\phi-\phi')}{2n}\right)F\,[kLa^{+}(\phi'-\phi)]\right.$$

$$\left.+\cot\left(\frac{\pi-(\phi-\phi')}{2n}\right)F\,[kL\,a^{-}(\phi-\phi')]\right]$$

$$\pm \left\{ \cot \left(\frac{\pi - (\phi - \phi')}{2n} \right) F[kL\, a^+ (\rho + \rho')] \right.$$

$$\left. \left. + \cot \left(\frac{\pi - (\phi + \phi')}{2n} \right) F[kLa^- (\phi + \phi')] \right\} \right] \tag{8}$$

The angular separation of the diffracted ray from RB, see the above figure, is

$$\delta = \pi - (\phi + \phi'), \tag{9}$$

and it is seen that the fourth cotangent in equation (8) becomes infinite as P crosses RB. We examine the limiting behaviour of cot ( ) F [ ] as this happens.

For small δ

$$\cot \left(\frac{\pi - (\phi + \phi')}{2n} \right) = \cot \frac{\delta}{2n} \sim \frac{2n}{\delta}$$

We now investigate the factor $F[kLa^- (\phi + \phi')]$.

The term

$$a^- (\phi + \phi') = a^- (\pi - \delta) = 2 \cos^2 \left(\frac{2n\pi N^- - (\pi - \delta)}{2} \right)$$

and N^- is the integer which most nearly satisfies

$$N^- = \frac{-(\pi - (\phi + \phi'))}{2\pi n}$$

$$= - \frac{\delta}{2\pi n}$$

Thus we use $N^- = 0$

Then $a^-(\phi + \phi') = 2 \cos^2 \left(-\frac{\pi}{2} + \frac{\delta}{2}\right)$

$$= 2 \sin^2 \frac{\delta}{2}$$

$$\simeq \frac{\delta^2}{2}$$

and we have

$$F[kLa^- (\phi + \phi')] \simeq F\left[kL \frac{\delta^2}{2}\right]$$

Taking the first term of the small argument expression gives

$$F\left[kL \frac{\delta^2}{2}\right] \simeq \sqrt{\frac{\pi kL}{2}}\, |\delta|\, e^{j \frac{\pi}{4}}$$

when δ is small.

The fourth term in the expression for $D_{s,h}$ is then

$$\text{Cot}\ (\ )\ F\ [\ ]\ \simeq\ \frac{2n}{\delta}\ \sqrt{\frac{\pi kL}{2}}|\delta|$$

$$= n\sqrt{2\pi kL}\ e^{j\frac{\pi}{4}}\ \text{sgn}\ \delta$$

As P crosses RB, sgn δ changes from +1 to -1 and the above term suffers a discontinuity of $-2n\sqrt{2\pi kL}$. It may be confirmed that, for the geometry of the figure, all the other terms are continuous. The discontinuity in D, see equation (8), is then

$$\begin{pmatrix} \Delta D_s \\ \Delta D_h \end{pmatrix} = \pm\sqrt{L}$$ ———————— Answer (b)

With $L = \dfrac{\rho\rho'}{\rho+\rho'}$,

we therefore have

$$\begin{pmatrix} \Delta D_s \\ \Delta D_h \end{pmatrix} = \pm\sqrt{\frac{\rho\rho'}{\rho+\rho'}}$$

and the discontinuity in the diffracted field is

$$\Delta u^d = A \begin{pmatrix} \Delta D_s \\ \Delta D_h \end{pmatrix} \frac{e^{-jk(\rho+\rho')}}{\sqrt{\rho\rho'}}$$

$$= \pm\ A\ \frac{e^{-jk(\rho\ +\ \rho')}}{\sqrt{\rho+\rho'}}$$

This discontinuity is equal in magnitude and opposite in sign to the discontinuity in the reflected field. The total field is thus free of discontinuities.
eqn (b)

An examination of the terms in eqn (8) will reveal that as the field crosses SB from region II to region III

$$\pi - (\phi - \phi') = 0$$

and the second cotangent factor is infinite while the F[ ] factor is zero. Once again it will be found that the diffracted field undergoes a discontinuity which cancels that of the direct field as the latter falls to zero on SB.

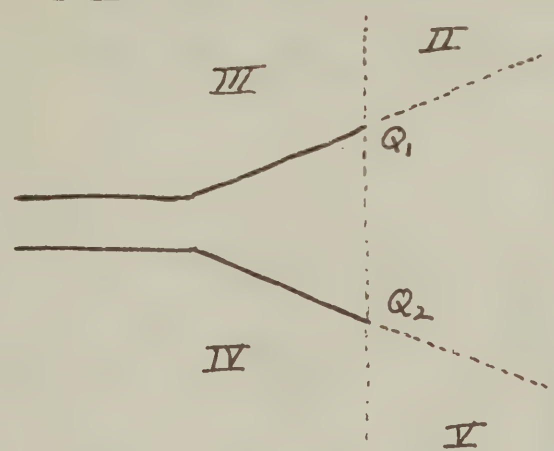
The other factors display similar behaviour in other geometries.

9.5-5

ϕ	$E_{\perp}^{d}$	$E_{\parallel}^{d}$
120°	0.28−j0.14	−0.14+j0.019
132°	0.22−j0.07	−0.08−j0.05
138°	0.11−j0.057	0.057−j0.075
180°	−0.045+j0.045	0.25−j0.066
222°	−0.51−j0.2	−0.28+j0.03
228°	−0.51+j0.22	−0.25−j0.42
260°	−0.32+j0.27	−0.03+j0.019

9.6-1 REFER TO FIG. 8-14

9.6-2



1) IN REGION I WE HAVE DIRECT RADIATION WHICH IS MUCH STRONGER THAN ANY OF THE RAYS IN FIG. 9-20 C.

2) IN REGION II THE DIFFRACTION FROM Q_1 AND Q_2 (PARTICULARLY Q_1) IS MUCH STRONGER THAN ANY OF THE RAYS IN FIG. 9-20 C.

3) IN MOVING ACROSS THE BOUNDARY FROM II TO III, Q_2 IS SUDDENLY SHADOWED, HENCE THE NEED FOR THE RAY FROM Q_2 TO Q_1.

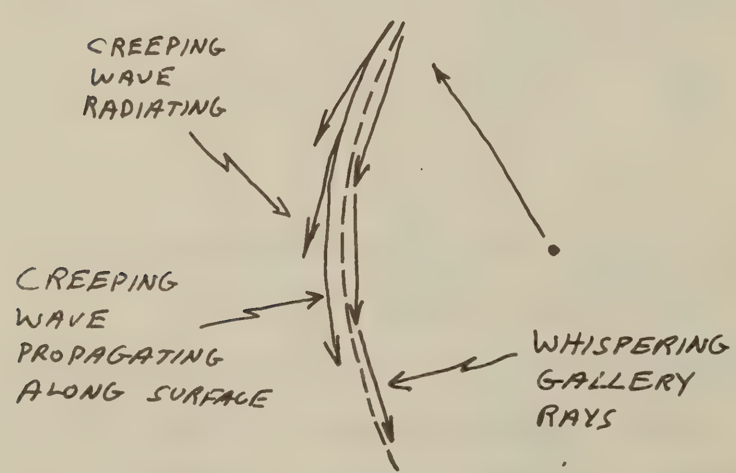
9.6-3

$$E^d_{1,2} = \frac{1}{2}\underbrace{\frac{e^{-j\beta\rho_E}}{\sqrt{\rho_E}}}_{\substack{\text{INCIDENT}\\ \text{FIELD}\\ \text{AT } Q_2}} \underbrace{D_{\perp 2}}_{\substack{\text{DIFF}\\ \text{COEF}\\ \text{AT } Q_2}} \underbrace{\frac{e^{-j\beta 2a}}{\sqrt{2a}}}_{\substack{\text{PATH}\\ \text{FROM } Q_2\\ \text{TO } Q_1}} \underbrace{D_{\perp 1}}_{\substack{\text{DIFF}\\ \text{COEF}\\ \text{AT } Q_1}} \underbrace{\frac{e^{-j(\beta r - a\sin\xi)}}{\sqrt{r}}}_{\substack{\text{PHASE AND}\\ \text{DISTANCE FROM}\\ Q_1 \text{ TO FAR FIELD}}}$$

9.7-1

THE FIELD IN THE FORWARD REGION OF THE REFLECTOR CONSISTS OF THE RADIATION FROM THE SURFACE OF THE REFLECTOR AS WELL AS THE DIFFRACTED FIELD, HENCE THE DIFFERENCE

9.7-2



9.7-3

$$E^d_{1,2}(P) = \underbrace{\frac{e^{-j\beta\rho_0}}{\sqrt{\rho_0}} f(\theta_E)}_{\substack{\text{INCIDENT}\\ \text{FIELD AT}\\ Q_2}} \underbrace{D_{\parallel 2}}_{\substack{\text{DIFF}\\ \text{COEF}\\ \text{AT } Q_2}} \underbrace{\frac{e^{-j\beta 2a}}{\sqrt{2a}}}_{\substack{\text{DISTANCE}\\ \text{FROM}\\ Q_2 \text{ TO } Q_1\\ \text{(AND PHASE)}}} \underbrace{D_{\parallel 1}}_{\substack{\text{DIFF}\\ \text{COEF}\\ \text{AT } Q_1}} \underbrace{\frac{e^{-j(\beta r - a\sin\xi)}}{\sqrt{r}}}_{\substack{\text{DISTANCE}\\ \text{AND PHASE}\\ \text{FROM } Q_1 \text{ TO}\\ \text{THE FAR FIELD}}}$$

9.7-4

FROM (8-26a)

$$E_\theta = j\beta \frac{e^{-j\beta r}}{2\pi r}\left(P_x \cos\phi + P_y \sin\phi\right)$$

HERE $\phi = 90°$. FURTHER THE PROBLEM IS 2-DIMENSIONAL,

$$E_\theta = j\beta \frac{e^{-j\beta r}}{2\pi \sqrt{r}} P_y$$

P_y IS GIVEN BY (8-18b) WHICH REDUCES TO

$$P_y = \int E_{ax}(y')\, e^{j\beta y' \sin\theta}\, dy'$$

BUT $E_{ax}(y') = \dfrac{f(\theta_s)}{\sqrt{\rho}}$ AND WE USE ξ INSTEAD OF θ TO AVOID AMBIGUITY PROBLEMS WITH θ,

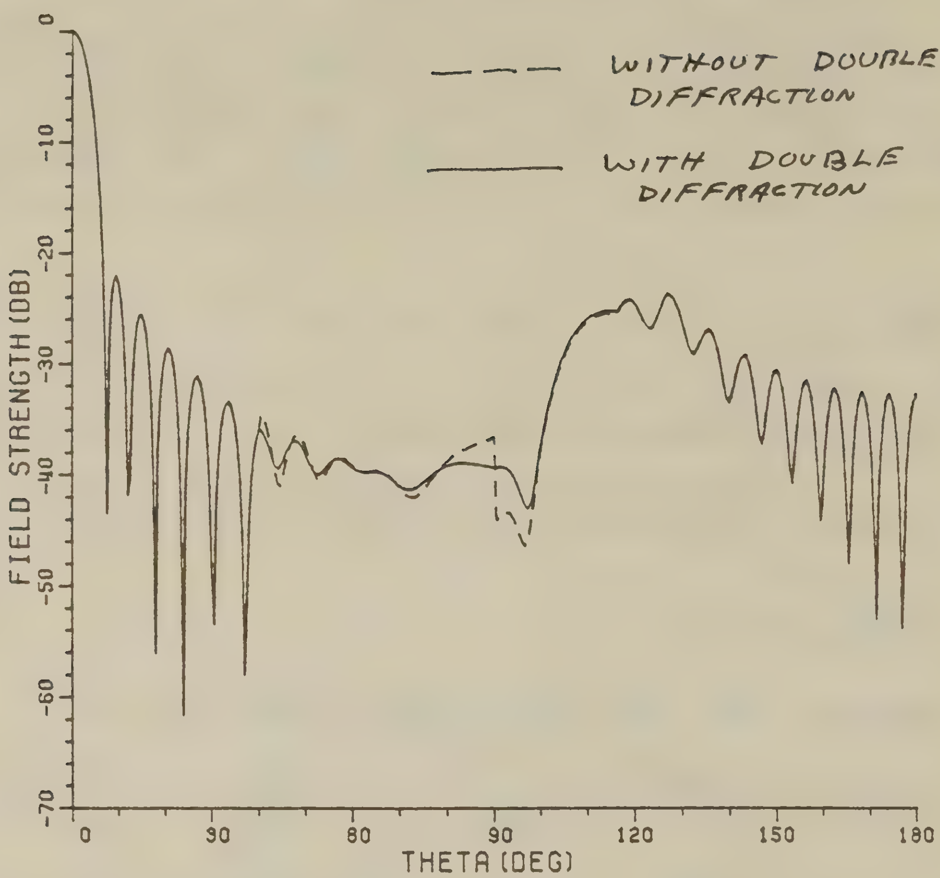
$$\therefore E_A(P) = j\beta \frac{e^{-j\beta r}}{2\pi\sqrt{r}} \int_{-a}^{a} \frac{f(\theta_s)}{\sqrt{\rho}}\, e^{j\beta y' \sin\xi}\, dy'$$

9.7-5

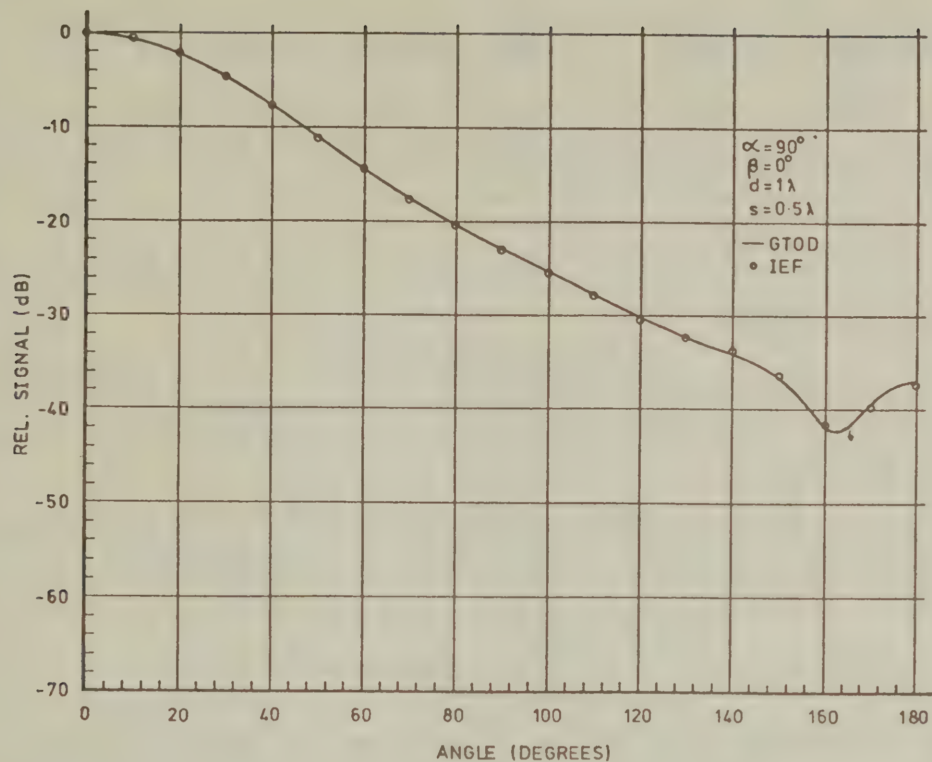
THE DISCONTINUITY AT 90° AND THE GREATER STRENGTH OF THE BACK LOBES IS DUE TO THE FACT THAT THE MAGNETIC LINE SOURCE RADIATES A FIELD ⊥ THE RIM OF THE PARABOLA.

(PATTERN IS ON NEXT PAGE.)

9.7-5 (CONT.)


WITHOUT DOUBLE DIFFRACTION
WITH DOUBLE DIFFRACTION
FIELD STRENGTH (DB)
0
-10
-20
-30
-40
-50
-60
-70
0
30
60
90
120
150
180
THETA (DEG)

9.7-6



A MORE "INTERESTING" PATTERN IS OBTAINED WHEN THE DIPOLE IS 1.5λ FROM THE APEX (SEE <u>ANTENNAS</u> BY KRAUS, PG. 332.)

9.8-1

AS $\theta \to 90^\circ$, $E_{\parallel}^{i} \to 0$, BUT $\left|\frac{\partial E^{i}}{\partial n}\right| > 0$. SO THE DIFFRACTED FIELD IS NOT ZERO NEAR 90°.

TO COMPARE VALUES, THE STUDENT MAY READ VALUES FROM FIG. 9-28.

9.8-2

a) FROM (9-46)

$$N_*^{r}(\rho, \phi+\phi') = e^{j\beta\rho\cos(\phi+\phi')} \xrightarrow[\phi'\to 0]{} e^{j\beta\rho\cos\phi}$$

FROM (9-47)

$$N_*^{i}(\rho, \phi-\phi') = e^{j\beta\rho\cos(\phi-\phi')} \xrightarrow[\phi'\to 0]{} e^{j\beta\rho\cos\phi}$$

$$\therefore N_*^{r} + N_*^{i} = 2e^{j\beta\rho\cos\phi} \quad \text{WHEN } \phi' = 0.$$

$\therefore$ THE INCIDENT AND REFLECTED FIELDS MERGE AND THE FACTOR OF 2 APPEARS. TO OFFSET THIS, WE MUST INJECT A VALUE OF $\frac{1}{2}$ WITH THE INCIDENT FIELD AS IN SEC. 9.8.

b) FROM 9-69 WHEN $\phi' = 0$

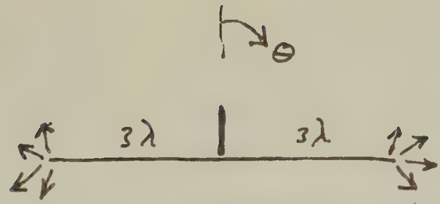
$$D_{\substack{\parallel \\ \perp}}(\phi, 0, 90^\circ) = \frac{e^{-j(\pi/4)}\sin(\pi/n)}{n\sqrt{2\pi\beta}}\left[\frac{1}{\cos\frac{\pi}{n} - \cos\frac{\phi}{n}} \mp \frac{1}{\cos\frac{\pi}{n} - \cos\frac{\phi}{n}}\right]$$

$$\therefore D_{\parallel} = 0$$

$$\therefore D_{\perp}(\phi, 0, 90^\circ) = 2\,\frac{e^{-j\pi/4}\sin(\pi/n)}{n\sqrt{2\pi\beta}}\;\frac{1}{\cos\frac{\pi}{n} - \cos\frac{\phi}{n}}$$

WHICH IMPLIES A FACTOR OF $\frac{1}{2}$ IS REQUIRED TO COMPENSATE.

9.9-1



UNDERNEATH THE GROUND PLANE WE HAVE

$$E_1^d(P) = V_{B_1}^r(\phi+\phi') + V_{B_1}^i(\phi-\phi')$$

$$\phi' = 0$$

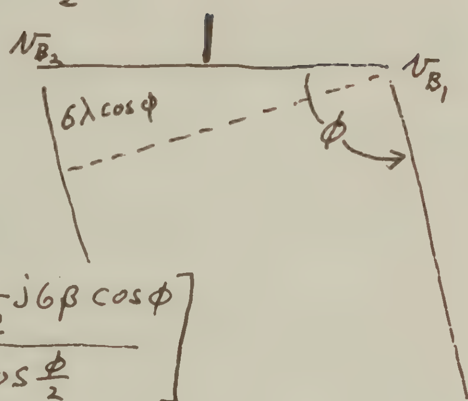
$$E_1^d(P) = 2V_{B_1}^i(\rho,\phi)$$

$\bullet P(\theta = 180^\circ)$

$$n = 2$$

$$E_1^d(P) = \frac{2e^{-j(\beta\rho-\pi/4)}}{\sqrt{2\pi\beta\rho}}\ \frac{\frac{1}{2}\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}-\cos\frac{\phi}{2}}$$

$$E_1^d(P) = \frac{e^{-j(\beta\rho-\pi/4)}}{\sqrt{2\pi\beta\rho}}\ \frac{1}{-\cos\frac{\phi}{2}}$$



$$E^d(P) = E_1^d(P) - e^{-j6\beta\cos\phi}E_2^d(P)$$

$$\therefore E^d(P) = \frac{e^{-j(\beta\rho-\pi/4)}}{\sqrt{2\pi\beta\rho}}\left[\frac{1}{\cos\frac{\phi}{2}} - \frac{e^{-j6\beta\cos\phi}}{\cos\frac{\phi}{2}}\right]$$

b) DIFFRACTED FIELD MUST BE ZERO AT $\theta = 180^\circ$, OR $\phi = 90^\circ$ BECAUSE THE DIFFRACTED FIELDS EXACTLY CANCEL AS SHOWN IN THE TOP SKETCH.

NOTE: $\theta + \phi = 3\pi/2$; $\phi = 3\pi/2 - \theta$.

9.10-1

FROM (9-102) $E_z = \dfrac{-\beta^2 I^e}{4\omega\epsilon} H_0^{(2)}(\beta\rho)$

OR WHEN $\beta\rho$ IS LARGE, $E_z = \eta\beta I^e \dfrac{e^{j(\pi/4)}}{2\sqrt{2\pi\beta\rho}} e^{-j\beta\rho}$

FROM (9-105)
$$E_z = D_{\parallel}(L,\phi,\phi') E_z^i \frac{e^{-j\beta\rho}}{\sqrt{\rho}}$$

$$\therefore \eta\beta I^e \frac{e^{j(\pi/4)}}{2\sqrt{2\pi\beta\rho}} e^{-j\beta\rho} = D_{\parallel}(L,\phi,\phi') E_z^i \frac{e^{-j\beta\rho}}{\sqrt{\rho}}$$

$$I^e = D_{\parallel}(L,\phi,\phi') E_z^i \frac{2\sqrt{2\pi\beta\rho}}{\eta\beta\sqrt{\rho}} e^{-j\pi/4}$$

$$I^e = \frac{-2j}{\eta\beta} E_z^i D_{\parallel}(\phi,\phi';\tfrac{\pi}{2})\sqrt{2\pi\beta}\, e^{j(\pi/4)} \qquad (9-107)$$

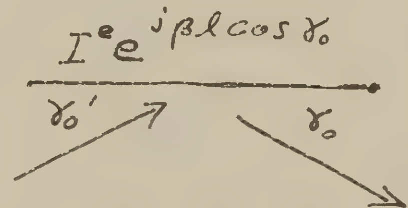
$$H_z = -\frac{\beta}{\eta} I^m \frac{e^{j(\pi/4)}}{2\sqrt{2\pi\beta\rho}} e^{-j\beta\rho} \qquad (9-104)$$

$$H_z = D_{\perp}(L,\phi,\phi') H_z^i \frac{e^{-j\beta\rho}}{\sqrt{\rho}} \qquad (9-106)$$

$$\therefore -\frac{\beta}{\eta} I^m \frac{e^{j(\pi/4)}}{2\sqrt{2\pi\beta\rho}} e^{-j\beta\rho} = D_{\perp}(L,\phi,\phi') H_z^i \frac{e^{-j\beta\rho}}{\sqrt{\rho}}$$

$$I^m = \frac{-2j\eta}{\beta} H_z^i D_{\perp}(\phi,\phi';\tfrac{\pi}{2})\sqrt{2\pi\beta}\, e^{j(\pi/4)} \qquad (9-108)$$

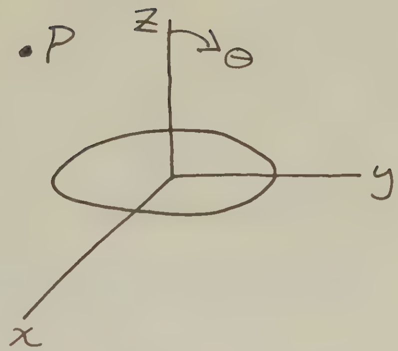
9.10-2



REPLACE I^e IN 9.10-1 WITH $I^e e^{j\beta l\cos\gamma_0}$

AND REPLACE I^m WITH $I^m e^{j\beta l\cos\gamma_0}$

9,10-3



$$E^d(P) = \int_0^{2\pi} N_B(\rho,\phi) \overbrace{\cos(\phi-\phi')}^{\text{POLARIZATION DEPENDENCE}} \cdot e^{j\beta r \sin\theta \cos(\phi-\phi')} d\phi'$$

$$E^d(P) = N_B(\rho,\phi) \int_0^{2\pi} \cos(\phi-\phi')\, e^{j\beta r \sin\theta \cos(\phi-\phi')} d\phi'$$

$$E^d(P) = N_B(\rho,\phi)\, 2\pi j\, J_1(\beta r \sin\theta)$$

$$E^d(P) = \frac{e^{-j(\beta\rho + \pi/4)}}{\sqrt{2\pi\beta\rho}} \; \frac{1}{-\cos\left(\frac{\theta+\pi/2}{2}\right)} \; 2\pi j\, J_1(6\pi \sin\theta)$$

9.11 - 1

AT CORNER A:

$$C_1 = C_3 T_{CA} + C_4 R_{BA} + V_1$$
$$C_6 = C_4 T_{BA} + C_3 R_{CA} + V_6$$

AT CORNER B:

$$C_2 = C_1 T_{AB} + C_5 R_{CB} + V_2$$
$$C_4 = C_5 T_{CB} + C_1 R_{AB} + V_4$$

AT CORNER C:

$$C_3 = C_2 T_{BC} + C_6 R_{AC} + V_3$$
$$C_5 = C_6 T_{AC} + C_2 R_{BC} + V_5$$

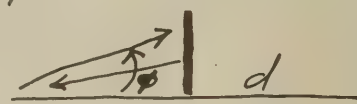
EXPRESSIONS FOR R, T AND V MAY BE OBTAINED DIRECTLY FROM THE TEXT PGS 497, 498.

9.13-1

IN A MANNER SIMILAR TO (9-98), (9-99) WE CAN WRITE THE FOLLOWING FOR ONE DIFFRACTION POINT

$$\bar{E}^d(z) = \hat{\theta}\, E_\perp^i(Q)\, D_\perp(L,\phi,\phi')\sqrt{d}\ \frac{e^{-j\beta r_1}}{r_1}$$

WHERE $E_\perp^i(Q) = \hat{z}\, E_0 \dfrac{e^{-j\beta d}}{d} = \hat{z}\, E_\perp^i(Q)$



$$\therefore\ Z^3_{mn} = 4\, E^d(z_m)\cos\phi$$

(THE 4 ARISES BECAUSE THERE ARE 4 EDGES)

9.13-2

FROM (9-108)

$$I^m = -\frac{2j\eta}{\beta} H_z^i\, D_\perp(\phi,\phi';\pi/2)\sqrt{2\pi\beta}\ e^{j(\pi/4)}$$

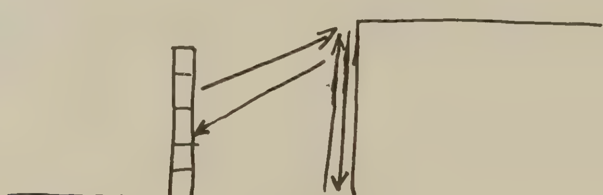
$$E_\theta^i = j\eta H_z^i$$

$$I^m = -2\, E_\theta^i\, D_\perp(\phi,\phi';\tfrac{\pi}{2})\sqrt{\tfrac{2\pi}{\beta}}\ e^{j\pi/4}$$

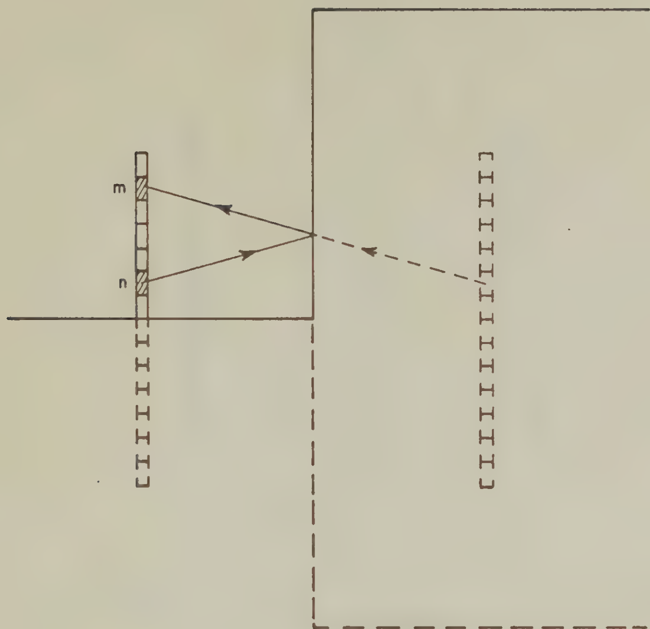
$$I^m = -2\, E_\theta^i\, D_\perp(\phi,\phi';\tfrac{\pi}{2})\sqrt{\lambda}\ e^{j\pi/4}$$

9.13-3

WE IGNORE ALL RAYS THAT WOULD INVOLVE MORE THAN ONE DIFFRACTION, FOR EXAMPLE!

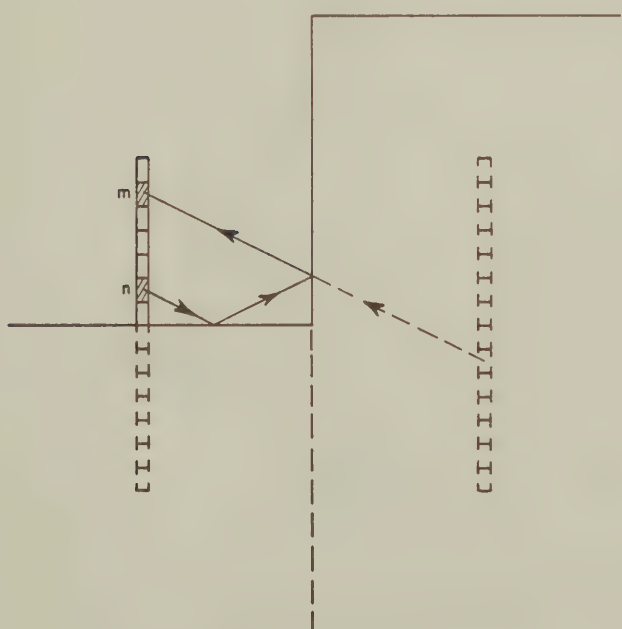


9.13-3 (CONT)



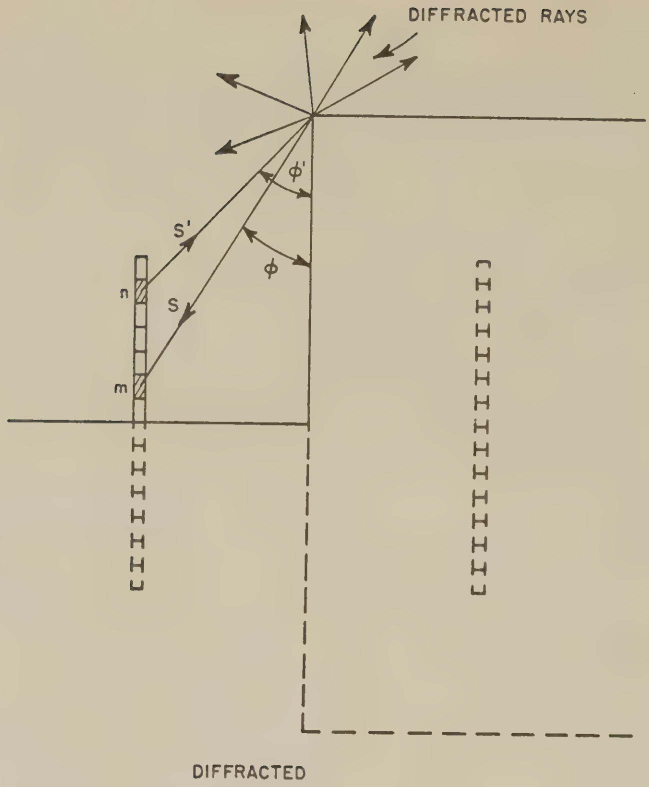
SINGLY REFLECTED

--Segmented monopole near a conducting step showing a ray that is singly reflected.

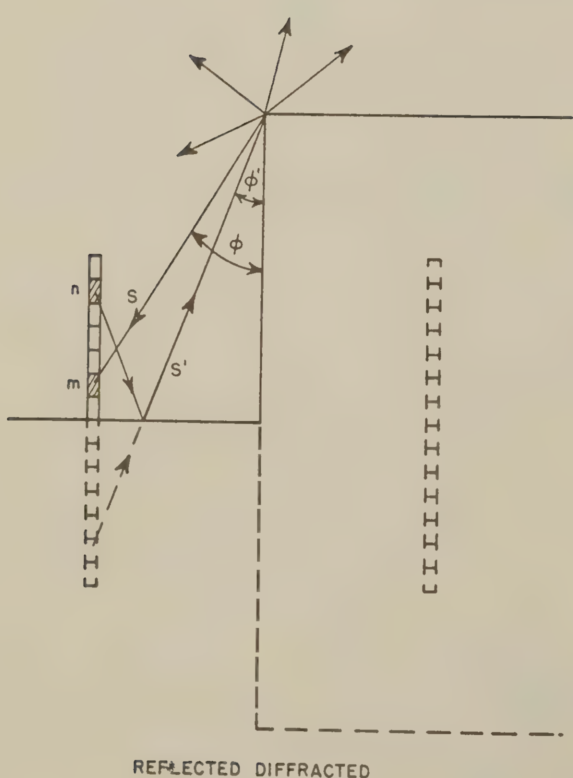


DOUBLY REFLECTED

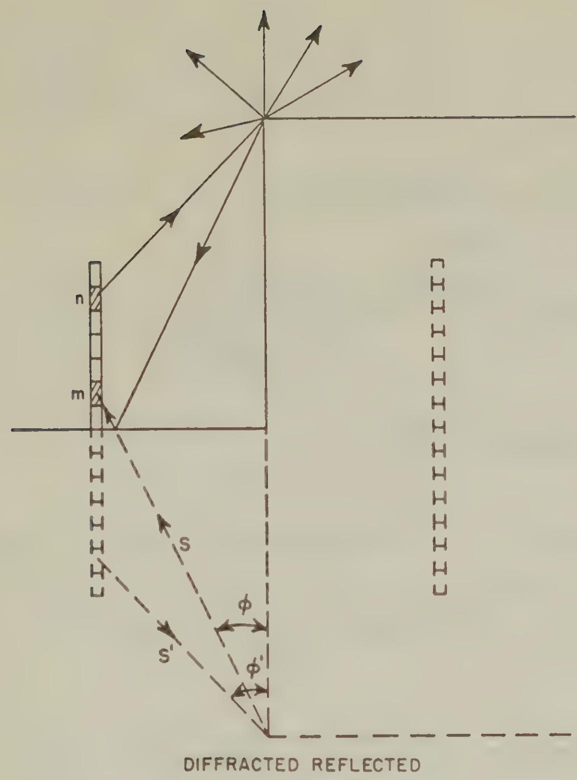
--Segmented monopole near a conducting step showing a ray that is doubly reflected.



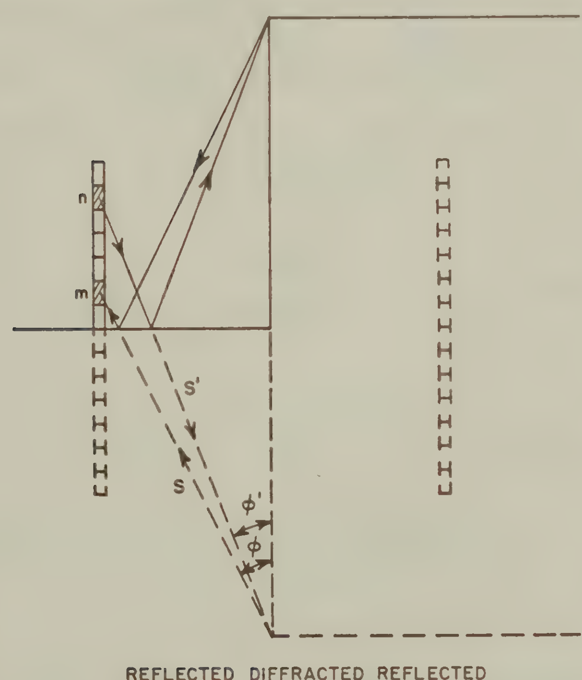
--Segmented monopole near a conducting step showing a ray that is diffracted.



--Segmented monopole near a conducting step showing a ray that is reflected and then diffracted.

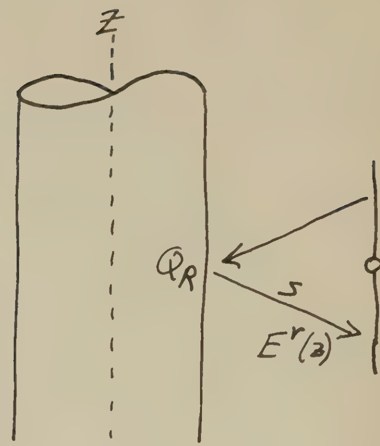


--Segmented monopole near a conducting step showing a ray that is diffracted and then reflected.



--Segmented monopole near a conducting step showing a ray that is reflected, then diffracted, and then reflected again.

9.13-4



$$Z^g_{mn} = E^r(z)$$

IF A POINT-MATCHING SOLUTION IS USED.

$$E^r(z) = E^i(Q_R)\ R \sqrt{\frac{\rho_1^r \rho_2^r}{(\rho_1^r+s)(\rho_2^r+s)}}$$

WHERE HERE $R = -1$.

IF A PWS (GALERKIN) SOLUTION IS USED,

$$Z^g_{mn} = \int E^r_m(z) \cdot I_n(z)\, dz$$

(SEE IEEE TRANS AP-28, NOV. 1980, 831-839)

ρ_1^r AND ρ_2^r ARE THE PRINCIPAL RADII OF CURVATURE OF THE REFLECTED WAVEFRONT AT THE REFLECTION POINT Q_R.

CHAPTER 10

10.2-1

$$g \to f \qquad t \to w \qquad G \to i \qquad \omega \to 2\pi s$$

Then

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega \to \frac{1}{2\pi}\int_{-\infty}^{\infty} i(s)e^{j2\pi s w}\, d(2\pi s)$$

or

$$f(w) = \int_{-\infty}^{\infty} i(s)e^{j2\pi s w}ds \qquad (10\text{-}6)$$

And

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt \to i(s) = \int_{-\infty}^{\infty} f(w)e^{-j2\pi s w}dw \qquad (10\text{-}7)$$

10.2-2

Uniform current $\quad i(s) = \begin{cases} I_o & |s| \le L/2\lambda \\ 0 & |s| > L/2\lambda \end{cases}$

From (10-6)

$$f(w) = \int_{-\infty}^{\infty} i(s)e^{j2\pi s w}dw = \int_{-L/2\lambda}^{L/2\lambda} I_o e^{j2\pi s w}dw = I_o\left[\frac{e^{j2\pi s w}}{j2\pi w}\right]_{-L/2\lambda}^{L/2\lambda}$$

$$= I\,\frac{e^{j2\pi\frac{L}{2\lambda}w} - e^{-j2\pi\frac{L}{2\lambda}w}}{j2\pi w} = I_o\,\frac{2j\sin(\pi\frac{L}{\lambda}w)}{2j\pi w} = \boxed{I_o\frac{L}{\lambda}\,\frac{\sin(\pi\frac{L}{\lambda}w)}{\pi\frac{L}{\lambda}w}}$$

10.2-3

Sector pattern $\quad f_d(w) = \begin{cases} 1 & |w| \le c \\ 0 & |w| > c \end{cases} \qquad (10\text{-}11b)$

From (10-8)

$$i_d(s) = \int_{-\infty}^{\infty} f_d(w)e^{-j2\pi s w}dw = \int_{-c}^{c} 1\, e^{-j2\pi s w}dw = \frac{e^{-j\pi s 2c} - e^{+j\pi s 2c}}{-j2\pi s}$$

$$= \frac{-2j\sin(\pi s 2c)}{-j2\pi s} = \boxed{2c\,\frac{\sin(\pi s 2c)}{\pi s 2c}} \qquad (10\text{-}12)$$

10.2-4 (a)

Fourier transform synthesis of a sector pattern

$$i(s) = 2c\,\frac{\sin(2\pi c s)}{2\pi c s} \qquad |s| \le \frac{L}{2\lambda} \qquad (10\text{-}12)$$

10.2-4 (con't)

$$f(w)=\int_{-\infty}^{\infty} i(s)\,e^{j2\pi sw}\,ds=\int_{-\frac{L}{2\lambda}}^{\frac{L}{2\lambda}} \frac{e^{j2\pi cs}-e^{-j2\pi cs}}{2j\pi s}\,e^{j2\pi ws}\,ds$$

$$=\frac{1}{2j\pi}\int_{-\frac{L}{2\lambda}}^{\frac{L}{2\lambda}}\left[\frac{e^{j2\pi(c+w)s}}{s}-\frac{e^{j2\pi(-c+w)s}}{s}\right]ds$$

The first term is

$$\frac{1}{2j\pi}\int_{-\frac{L}{2\lambda}}^{\frac{L}{2\lambda}}\frac{e^{j2\pi(c+w)s}}{s}\,ds=\frac{1}{2j\pi}\int_{-\frac{L}{2\lambda}}^{\frac{L}{2\lambda}}\left[\frac{\cos 2\pi(c+w)s}{s}+j\,\frac{\sin 2\pi(c+w)s}{s}\right]ds$$

$$=\frac{1}{2j\pi}\,2j\int_{0}^{\frac{L}{2\lambda}}\frac{\sin 2\pi(c+w)s}{s}\,ds$$

$t=2\pi(c+w)s$

$dt=2\pi(c+w)ds$

$$=\frac{1}{\pi}\int_{0}^{2\pi(c+w)\frac{L}{2\lambda}}\frac{\sin t}{t/2\pi(c+w)}\,\frac{dt}{2\pi(c+w)}$$

$$=\frac{1}{\pi}\int_{0}^{\frac{L}{\lambda}\pi(c+w)}\frac{\sin t}{t}\,dt=\frac{1}{\pi}Si\left[\frac{L}{\lambda}\pi(c+w)\right]\quad \text{using (F-13)}$$

Similarly for the second term

$$-\frac{1}{2j\pi}\int_{-\frac{L}{2\lambda}}^{\frac{L}{2\lambda}}\frac{e^{j2\pi(-c+w)}}{s}\,ds=-\frac{1}{\pi}Si\left[\frac{L}{\lambda}\pi(-c+w)\right]$$

So

$$f(w)=\frac{1}{\pi}\left[Si\left(\frac{L}{\lambda}\pi(c+w)\right)-Si\left(\frac{L}{\lambda}\pi(-c+w)\right)\right]\quad \text{which is (10-13)}$$

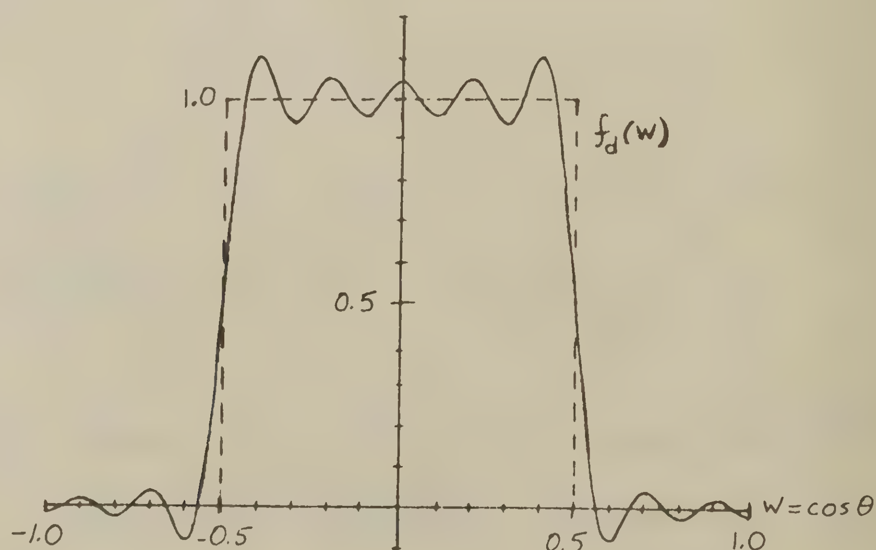
(b)

Figure 10-1a should be →

SLL = −21.9 dB

R = 0.83 dB

T = 0.0893



10.2-5

From (10-6) and (10-14)

$$f_n(w) = \int_{-\infty}^{\infty} i_n(s) e^{j2\pi ws} ds = \int_{-\frac{L}{2\lambda}}^{\frac{L}{2\lambda}} \frac{a_n}{L/\lambda} e^{-j2\pi w_n s} e^{j2\pi ws} ds$$

$$= \frac{a_n}{L/\lambda} \frac{e^{j2\pi(w-w_n)\frac{L}{2\lambda}} - e^{j2\pi(w-w_n)\frac{L}{2\lambda}}}{j2\pi(w-w_n)} = \frac{a_n}{L/\lambda} \frac{2j\sin[2\pi(w-w_n)\frac{L}{2\lambda}]}{j2\pi(w-w_n)}$$

$$= a_n \frac{\sin[\pi\frac{L}{\lambda}(w-w_n)]}{\pi\frac{L}{\lambda}(w-w_n)} = a_n \, Sa[\pi\tfrac{L}{\lambda}(w-w_n)]$$ which is (10-15).

10.2-6

(a) $L = 5\lambda \quad w_n = \frac{n}{L/\lambda} = \frac{n}{5} = 0.2n \qquad a_n = f_d(w=w_n)$

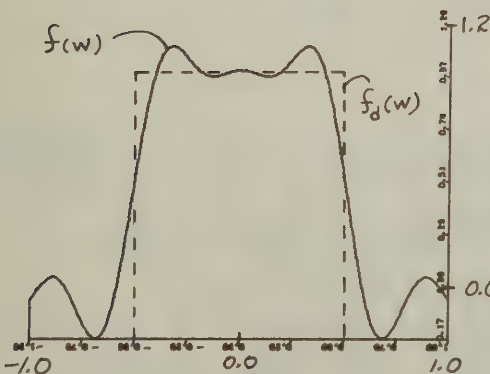
So

n	w_n	a_n	n	w_n	a_n
0	0	1.0	±3	±0.6	0
±1	±0.2	1.0	±4	±0.8	0
±2	±0.4	1.0	±5	±1.0	0

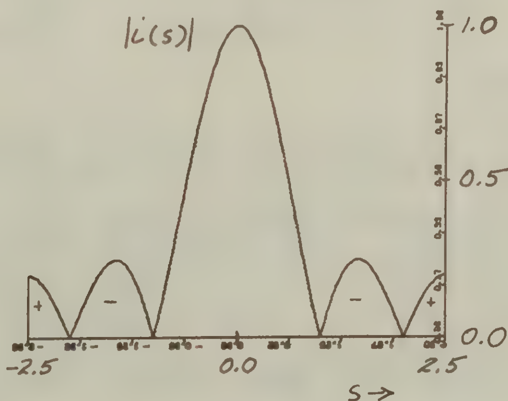
```
SPAP - SAMPLING PATTERN ANTENNA PROGRAM
THIS PATTERN IS EXPRESSIBLE AS A SUM OF SA FUNCTIONS
APERTURE LENGTH= LEN=   5.0000 WAVELENGTHS
NUMBER OF SAMPLE POINTS= NP=     5
    I     WN(I)      A(I)
    1   0.400000   1.000000
    2   0.200000   1.000000
    3   0.0        1.000000
    4  -0.200000   1.000000
    5  -0.400000   1.000000
```

(b)

Pattern



Current



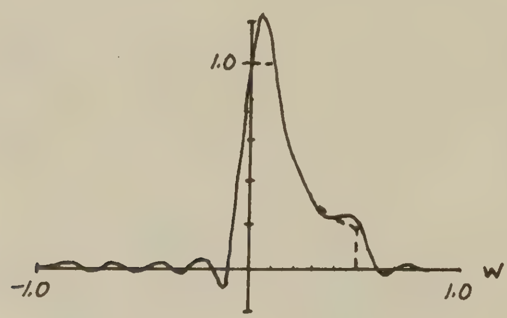
10.2-7

$$f_d(w) = \begin{cases} 1 & 0 \le w \le 0.1 \\ 0.1/w & 0.1 \le w \le 0.5 \\ 0 & \text{elsewhere} \end{cases} \qquad L = 10\lambda$$

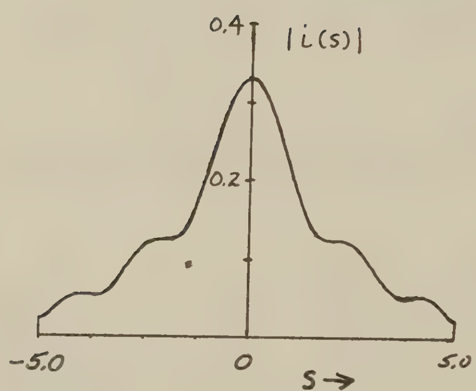
(a)

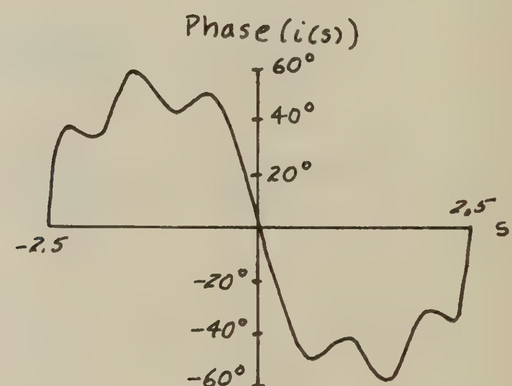
n	$w_n = \frac{n}{L/\lambda} = 0.1n$	$a_n = f_d(w=w_n)$
0	0	1.0 (could use 0.5 here too)
1	0.1	1.0
2	0.2	0.1/0.2 = 0.500
3	0.3	0.1/0.3 = 0.333
4	0.4	0.1/0.4 = 0.250
5	0.5	0.1/0.5 = 0.200

rest are zero sample values



(b)





10.3-1

If the desired pattern $f_d(w)$ satisfies the Dirichlet sufficiency conditions that in a finite interval the function is finite and has a finite number of extrema and a finite number of discontinuities, then the Fourier series representation converges to $f_d(w)$ wherever $f_d(w)$ is continuous inside the interval, and at each point of the discontinuity it will converge to the average of the right and left hand limits. In the interval $-\pi < \Psi < \pi$

10.3-1 (con't)

$$f_d(w) = \sum_{m=-\infty}^{\infty} b_m e^{jm\psi} \qquad \text{where} \quad \psi = \beta d \cos\theta$$

for an array on the z-axis. The Fourier coefficients are

$$b_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_d(\psi) e^{-jm\psi} d\psi = \frac{d}{\lambda}\int_{-\lambda/2d}^{\lambda/2d} f_d(w) e^{-j2\pi m \frac{d}{\lambda} w} dw \qquad (10\text{-}25)$$

Or we can write

$$f_d(\psi) = \sum_{m=1}^{\infty} b_{-m} e^{-j\frac{2m-1}{2}\psi} + \sum_{m=1}^{\infty} b_m e^{j\frac{2m-1}{2}\psi}$$

$$b_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_d(\psi) e^{-j\frac{2m-1}{2}\psi} d\psi \qquad m>0$$

$$b_{-m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_d(\psi) e^{j\frac{2m-1}{2}\psi} d\psi \qquad -m<0$$

10.3-2

$$f_d(w) = \sum_{m=-\infty}^{\infty} b_m e^{jm2\pi\frac{d}{\lambda}w} \qquad (10\text{-}24)$$

Integrating both sides

$$\int_{-\lambda/2d}^{\lambda/2d} f_d(w) e^{-j2\pi n\frac{d}{\lambda}w} dw = \sum_{m=-\infty}^{\infty} b_m \int_{-\lambda/2d}^{\lambda/2d} e^{j2\pi(m-n)\frac{d}{\lambda}w} dw$$

$$= \sum_{m=-\infty}^{\infty} b_m \frac{e^{j2\pi(m-n)\frac{d}{\lambda}\frac{\lambda}{2d}} - e^{-j2\pi(m-n)\frac{d}{\lambda}\frac{\lambda}{2d}}}{j2\pi(m-n)\frac{d}{\lambda}} = \sum_{m=-\infty}^{\infty} b_m \frac{2j\sin[\pi(m-n)]}{2j\pi(m-n)\frac{d}{\lambda}}$$

$$= \frac{\lambda}{d}\sum_{m=-\infty}^{\infty} b_m \frac{\sin[\pi(m-n)]}{\pi(m-n)} = \frac{\lambda}{d} b_n$$

$$\therefore\ b_m = \frac{d}{\lambda}\int_{-\lambda/2d}^{\lambda/2d} f_d(w) e^{-j2\pi m\frac{d}{\lambda}w} dw \qquad (10\text{-}25)$$

10.3-3

Using (10-11) in (10-28) for $m \geq 1$

$$i_m = b_m = \frac{d}{\lambda}\int_{-\frac{\lambda}{2d}}^{\frac{\lambda}{2d}} f_d(w) e^{-j\pi(2m-1)\frac{d}{\lambda}w} dw = \frac{d}{\lambda}\int_{-c}^{c} e^{-j\pi(2m-1)\frac{d}{\lambda}w} dw \quad \text{for } c<\frac{\lambda}{2d}$$

$$= \frac{d}{\lambda}\ \frac{e^{-j\pi(2m-1)\frac{d}{\lambda}c} - e^{j\pi(2m-1)\frac{d}{\lambda}c}}{-j\pi(2m-1)\frac{d}{\lambda}} = 2\frac{d}{\lambda}c\ \mathrm{Sa}\left[\pi(2m-1)\frac{d}{\lambda}c\right]$$

which is (10-29)

10.3-3 (con't)

For $-m \le -1$

$$i_{-m} = b_{-m} = \frac{d}{\lambda}\int_{-c}^{c} e^{j\pi(2m-1)\frac{d}{\lambda}w}\,dw = \frac{d}{\lambda}\,\frac{e^{j\pi(2m-1)\frac{d}{\lambda}c} - e^{-j\pi(2m-1)\frac{d}{\lambda}c}}{j\pi(2m-1)\frac{d}{\lambda}c}$$

$$= 2\frac{d}{\lambda}c\, Sa\left[\pi(2m-1)\frac{d}{\lambda}c\right] \quad \text{which is (10-29).}$$

10.3-4

$P=20$, $d=0.6\lambda$, Fourier series, sector pattern ($c=0.5$)

(a) From (10-29)

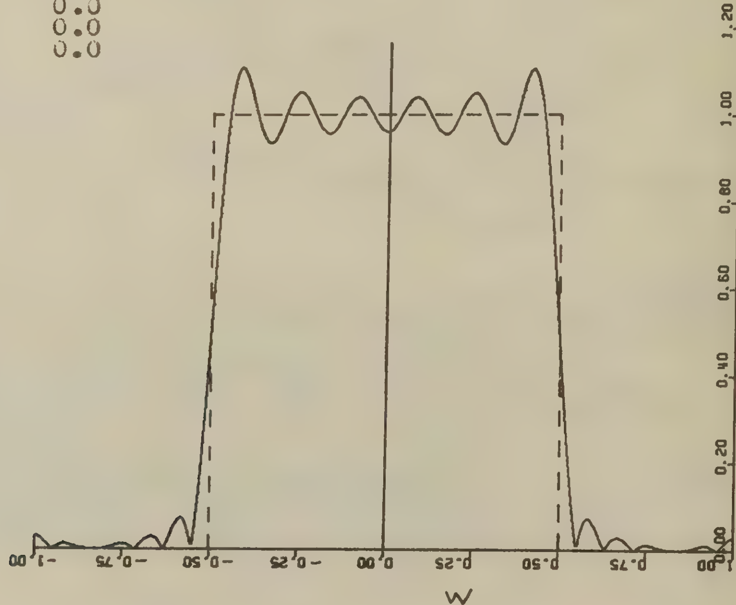
$i_m = i_{-m} = 2\frac{d}{\lambda}c\, Sa\left[\pi(2m-1)\frac{d}{\lambda}c\right]$ for $1 \le m \le N$ if $c < \frac{\lambda}{2d}$ which is true for $d=0.6\lambda$ since $0.5 < \frac{\lambda}{2(0.6\lambda)} = 0.833$.

$$i_m = i_{-m} = 2(0.6)(0.5)\, Sa\left[\pi(2m-1)\,0.6(0.5)\right] = 0.6\, Sa\left[(2m-1)0.3\pi\right]$$

THE NUMBER OF ARRAY ELEMENTS = 20

I	S(I)	A(I)	PHASE(I)
1	5.7000	-0.027110	0.0
2	5.1000	-0.011570	0.0
3	4.5000	0.042440	0.0
4	3.9000	-0.015130	0.0
5	3.3000	-0.046820	0.0
6	2.7000	0.057230	0.0
7	2.1000	0.028100	0.0
8	1.5000	-0.127320	0.0
9	0.9000	0.065570	0.0
10	0.3000	0.515040	0.0
11	-0.3000	0.515040	0.0
12	-0.9000	0.065570	0.0
13	-1.5000	-0.127320	0.0
14	-2.1000	0.028100	0.0
15	-2.7000	0.057230	0.0
16	-3.3000	-0.046820	0.0
17	-3.9000	-0.015130	0.0
18	-4.5000	0.042440	0.0
19	-5.1000	-0.011570	0.0
20	-5.7000	-0.027110	0.0

(b)



<u>10.3-5</u> $P=10$, $d=0.5\lambda$, Fourier series, sector pattern $(c=0.5)$

(a)

$$i_m = i_{-m} = 2\frac{d}{\lambda}c\,Sa\left[\pi(2m-1)\frac{d}{\lambda}c\right] \text{ if } c<\frac{\lambda}{2d}\text{, which is true}$$

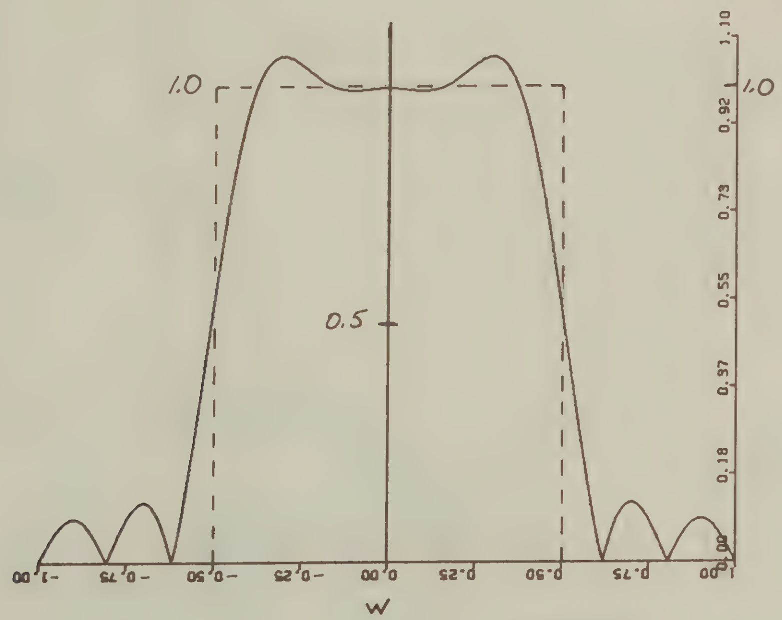
for $d=0.5$ since $0.5<\frac{\lambda}{2(0.5\lambda)}=1$.

So

$$i_m = i_{-m} = 2(0.5)(0.5)\,Sa[\pi(2m-1)(0.5)(0.5)] = \frac{1}{2}Sa\left[\frac{\pi}{4}(2m-1)\right]$$

THE NUMBER OF ARRAY ELEMENTS = 10

I	S(I)	A(I)	PHASE(I)
1	2.2500	0.050020	0.0
2	1.7500	-0.064310	0.0
3	1.2500	-0.090030	0.0
4	0.7500	0.150050	0.0
5	0.2500	0.450150	0.0
6	-0.2500	0.450150	0.0
7	-0.7500	0.150050	0.0
8	-1.2500	-0.090030	0.0
9	-1.7500	-0.064310	0.0
10	-2.2500	0.050020	0.0



<u>10.3-6</u>

$P=21$, $d=0.5\lambda$, Fourier series, sector pattern $(c=0.5)$

Using (10-25)

$$i_m = b_m = \frac{d}{\lambda}\int_{-\frac{\lambda}{2d}}^{\frac{\lambda}{2d}} f_d(w)e^{-j2\pi m\frac{d}{\lambda}w}dw = \frac{d}{\lambda}\int_{-c}^{c} 1\, e^{-j2\pi m\frac{d}{\lambda}w}dw$$

$$=\frac{d}{\lambda}\frac{e^{-j2\pi m\frac{d}{\lambda}c}-e^{j2\pi m\frac{d}{\lambda}c}}{-j2\pi m\frac{d}{\lambda}} = \frac{d}{\lambda}\frac{2j\sin(2m\pi\frac{d}{\lambda}c)}{j2\pi m\frac{d}{\lambda}c} = 2\frac{d}{\lambda}c\frac{\sin(2\pi m\frac{d}{\lambda}c)}{2\pi m\frac{d}{\lambda}c}$$

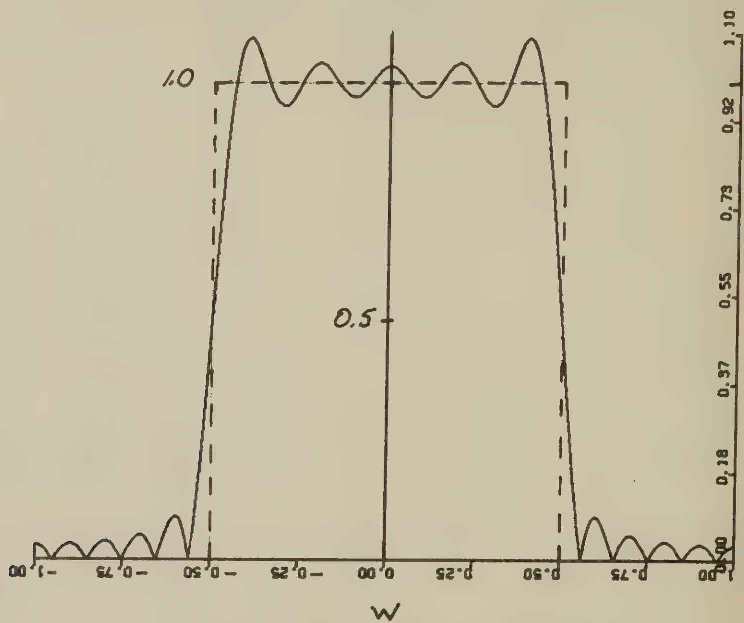
10.3-6 (con't)

Evaluating for $d=0.5\lambda$ and $c=0.5$

$$i_m = \frac{1}{2}\,\frac{\sin(m\pi/2)}{m\pi/2} \qquad \text{which is zero for } m \text{ even}$$

THE NUMBER OF ARRAY ELEMENTS = 11 (10 elements have zero current)

I	S(I)	A(I)	PHASE(I)
1	4.5000	0.035370	0.0
2	3.5000	-0.045470	0.0
3	2.5000	0.063660	0.0
4	1.5000	-0.106100	0.0
5	0.5000	0.318310	0.0
6	0.0	0.500000	0.0
7	-0.5000	0.318310	0.0
8	-1.5000	-0.106100	0.0
9	-2.5000	0.063660	0.0
10	-3.5000	-0.045470	0.0
11	-4.5000	0.035370	0.0



10.3-7

$P=20$, $d=0.5\lambda$, Fourier Series

$$f_d(w) = \begin{cases} 1 & 0 \le w \le 0.1 \\ 0.1/w & 0.1 \le w \le 0.5 \\ 0 & \text{elsewhere} \end{cases}$$

Using (10-28)

$$i_m = \frac{d}{\lambda}\int_{-\frac{\lambda}{2d}}^{\frac{\lambda}{2d}} f_d(w)\, e^{-j\pi(2m-1)\frac{d}{\lambda}w}\, dw \qquad m \geq 1$$

$$= \frac{1}{2}\int_0^{0.1} e^{-j\pi(2m-1)\frac{1}{2}w}\, dw + \frac{1}{2}\int_{0.1}^{0.5} \frac{0.1}{w}\, e^{-j\pi(2m-1)\frac{1}{2}w}\, dw$$

$$= \frac{1}{2}\,\frac{e^{-j\frac{\pi}{2}(2m-1)0.1}-1}{-j\frac{\pi}{2}(2m-1)} + 0.05\,\mathcal{J} = \frac{1}{2}e^{-j\frac{\pi}{4}(2m-1)0.1}\,\frac{-j2\sin\left(\frac{\pi}{4}(2m-1)0.1\right)}{-j\frac{\pi}{2}(2m-1)} + 0.05\,\mathcal{J}$$

10.3-7 (con't)

Let $t = \pi(2m-1)\frac{\pi}{2}$ $\quad dt = \frac{\pi}{2}(2m-1)dw$

So

$$\mathcal{J} = \int_{0.1}^{0.5}\left[\frac{\cos(\pi(2m-1)\frac{w}{2})}{w} - j\,\frac{\sin(\pi(2m-1)\frac{w}{2})}{w}\right]dw$$

$$= \int_{t_1}^{t_2}\left[\frac{\cos t}{t/\frac{\pi}{2}(2m-1)}\ \frac{dt}{\frac{\pi}{2}(2m-1)} - j\ \frac{\sin t}{t/\frac{\pi}{2}(2m-1)}\ \frac{dt}{\frac{\pi}{2}(2m-1)}\right] \qquad \begin{array}{l} t_1 = 0.1\frac{\pi}{2}(2m-1) \\ t_2 = 0.5\frac{\pi}{2}(2m-1)\end{array}$$

$$= \int_{t_1}^{t_2}\frac{\cos t}{t}dt - j\int_{t_1}^{t_2}\frac{\sin t}{t}dt = \int_{t_1}^{\infty}\frac{\cos t}{t}dt - \int_{t_2}^{\infty}\frac{\cos t}{t}dt - j\left[\int_0^{t_2}\frac{\sin t}{t}dt\right.$$

$$\left. - \int_0^{t_1}\frac{\sin t}{t}dt\right] = -Ci(t_1) - (-Ci(t_2)) - j\left[Si(t_2) - Si(t_1)\right]$$

$$= Ci(t_2) - Ci(t_1) - j\left[Si(t_2) - Si(t_1)\right] \qquad \text{using (F-13) and (F-14)}$$

Then

$$i_m = \frac{1}{2}e^{-j\frac{t_1}{2}}\,0.1\,\frac{\sin(t_1/2)}{t_1/2} + 0.05\,\mathcal{J}$$

$$= 0.05\left[e^{-j t_1/2}\ \frac{\sin t_1/2}{t_1/2} + Ci(t_2) - Ci(t_1) - j\,(Si(t_2) - Si(t_1))\right]$$

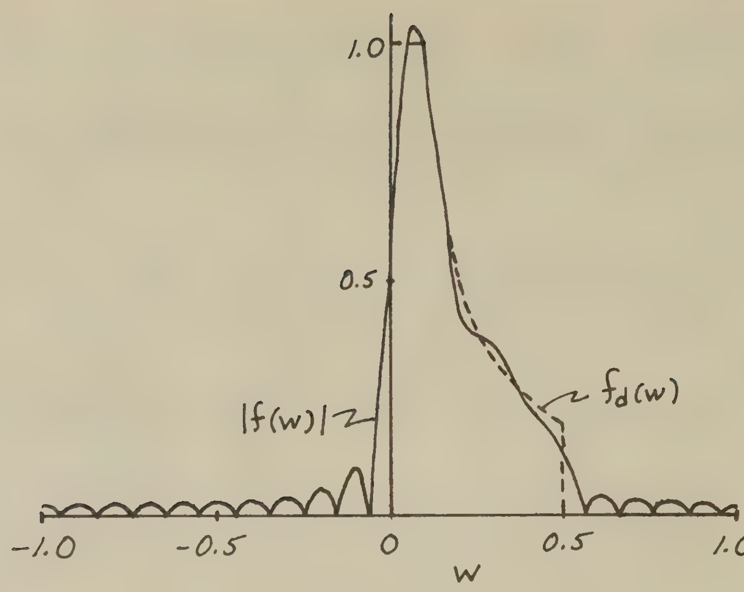
Similarly for $-m$

$$i_{-m} = 0.05\left[e^{j\,t_1/2}\ \frac{\sin t_1/2}{t_1/2} + Ci(t_2) - Ci(t_1) + j\,(Si(t_2) - Si(t_1))\right]$$

THE NUMBER OF ARRAY ELEMENTS = 20

I	S(I)	A(I)	PHASE(I)
1	-4.7500	0.021669	92.9700
2	-4.2500	0.020459	91.6100
3	-3.7500	0.025494	100.7300
4	-3.2500	0.036745	93.0700
5	-2.7500	0.043390	76.9700
6	-2.2500	0.042678	68.0900
7	-1.7500	0.048941	73.2000
8	-1.2500	0.075227	67.0600
9	-0.7500	0.107117	44.6600
10	-0.2500	0.127676	15.4600
11	0.2500	0.127676	-15.4600
12	0.7500	0.107117	-44.6600
13	1.2500	0.075227	-67.0600
14	1.7500	0.048941	-73.2000
15	2.2500	0.042678	-68.0900
16	2.7500	0.043390	-76.9700
17	3.2500	0.036745	-93.0700
18	3.7500	0.025494	-100.7300
19	4.2500	0.020459	-91.6100
20	4.7500	0.021669	-92.9700

10.3-7 (con't)



<u>10.3-8</u>

From (10-35) $i_m = \frac{1}{P} \sum_{n=-M}^{M} a_n e^{-j2\pi \frac{z_m}{\lambda} w_n}$ for a Woodward-Lawson array

(a) Using (10-20) for P odd

$$f(w) = \sum_{m=-N}^{N} i_m e^{j2\pi m \frac{d}{\lambda} w} = \frac{1}{P} \sum_{n=-M}^{M} a_n \sum_{m=-N}^{N} e^{j2\pi (m\frac{d}{\lambda} w - \frac{z_m}{\lambda} w_n)}$$

Using $z_m = md$ from (10-19), the sum on m is

$$\sum_{m=-N}^{N} e^{j2\pi m (w-w_n)\frac{d}{\lambda}} = e^{-j2\pi N (w-w_n)\frac{d}{\lambda}} \sum_{m=0}^{2N+1} e^{j2\pi m (w-w_n)\frac{d}{\lambda}}$$

$$= e^{-jN\psi} \frac{1-e^{j(2N+1)\psi}}{1-e^{j\psi}} \qquad \psi = 2\pi (w-w_n)\frac{d}{\lambda}$$

$$P = 2N+1$$

$$= e^{-jN\psi} \frac{e^{j\frac{P}{2}\psi}}{e^{j\psi/2}} \frac{e^{-j\frac{P}{2}\psi} - e^{j\frac{P}{2}\psi}}{e^{-j\psi/2} - e^{j\psi/2}} = e^{j(-N+\frac{P}{2}-\frac{1}{2})} \frac{\sin \frac{P}{2}\psi}{\sin \frac{\psi}{2}}$$

$$= \frac{\sin P\frac{\psi}{2}}{\sin \frac{\psi}{2}} \quad \text{since } -N+\frac{P}{2}-\frac{1}{2} = -N + \frac{2N+1}{2} - \frac{1}{2} = 0$$

So

$$f(w) = \sum_{n=-M}^{M} a_n \frac{\sin[P\, 2\pi (w-w_n)\frac{d}{\lambda}]}{P \sin[\frac{1}{2} 2\pi (w-w_n)\frac{d}{\lambda}]}$$ which is (10-32).

10.3-8 (con't)

(b) From (10-22) for P even

$$f(w)=\sum_{m=1}^{N}\left[i_{-m}e^{-j\pi(2m-1)\frac{d}{\lambda}w}+i_m e^{j\pi(2m-1)\frac{d}{\lambda}w}\right]$$

Substituting (10-35)

$$f(w)=\frac{1}{P}\sum_{m=1}^{N}\left[e^{-j\pi(2m-1)\frac{d}{\lambda}w}\sum_{n=-M}^{M}a_n e^{-j2\pi\frac{z_{-m}}{\lambda}w_n}\right.$$
$$\left.+e^{j\pi(2m-1)\frac{d}{\lambda}w}\sum_{n=-M}^{M}a_n e^{-j2\pi\frac{z_m}{\lambda}w_n}\right]$$

Interchanging sums and using

$$z_m=\frac{2m-1}{2}d \qquad z_{-m}=-\frac{2m-1}{2}d \qquad \psi=2\pi(w-w_n)\frac{d}{\lambda}\ ,$$

$$f(w)=\frac{1}{P}\sum_{n=-M}^{M}a_n\sum_{m=1}^{N}\left[e^{-j\frac{2m-1}{2}\psi}+e^{j\frac{2m-1}{2}\psi}\right]$$

The sum on m is

$$e^{j\frac{\psi}{2}}\sum_{m=1}^{N}e^{-jm\psi}+e^{-j\frac{\psi}{2}}\sum_{m=1}^{N}e^{jm\psi}$$
$$=e^{j\frac{\psi}{2}}e^{-j\psi}\sum_{m=1}^{N}e^{-j(m-1)\psi}+e^{-j\frac{\psi}{2}}e^{j\psi}\sum_{m=1}^{N}e^{j(m-1)\psi}$$
$$=e^{-j\frac{\psi}{2}}\frac{1-e^{-jN\psi}}{1-e^{-j\psi}}+e^{j\frac{\psi}{2}}\frac{1-e^{jN\psi}}{1-e^{j\psi}} \qquad \text{since } \sum_{m=1}^{N}r^{m-1}=\frac{1-r^N}{1-r}$$
$$=e^{-j\frac{\psi}{2}}\frac{e^{-j\frac{N}{2}\psi}}{e^{-j\psi/2}}\frac{e^{j\frac{N}{2}\psi}-e^{-j\frac{N}{2}\psi}}{e^{j\frac{\psi}{2}}-e^{-j\frac{\psi}{2}}}+e^{j\frac{\psi}{2}}\frac{e^{j\frac{N}{2}\psi}}{e^{j\psi/2}}\frac{e^{-j\frac{N}{2}\psi}-e^{j\frac{N}{2}\psi}}{e^{-j\frac{\psi}{2}}-e^{j\psi/2}}$$
$$=\left(e^{-j\frac{N}{2}\psi}+e^{j\frac{N}{2}\psi}\right)\frac{\sin N\frac{\psi}{2}}{\sin\frac{\psi}{2}}=\frac{2\cos N\frac{\psi}{2}\ \sin N\frac{\psi}{2}}{\sin\frac{\psi}{2}}$$
$$=\frac{\sin N\frac{\psi}{2}}{\sin\frac{\psi}{2}}$$

$$\therefore\ f(w)=\sum_{n=-M}^{M}a_n\frac{\sin\left[P\pi(w-w_n)\frac{d}{\lambda}\right]}{P\sin\left[\pi(w-w_n)\frac{d}{\lambda}\right]}\ \text{which is (10-32).}$$

10.3-9

$P=20, N=10, d=\frac{\lambda}{2}$, a_n and w_n are given in Table 10-2

First

$$z_m = \frac{2m-1}{2}d \qquad 1 \le m \le N \qquad z_{-m} = -\frac{2m-1}{2} \qquad -N \le -m \le -1$$

$$z_{\pm 1} = \pm\frac{1}{4}\lambda,\ z_{\pm 2} = \frac{3}{4}\lambda,\ z_{\pm 3} = \frac{5}{4}\lambda,\ z_{\pm 4} = \frac{7}{4}\lambda, \ldots, z_{\pm 10} = \frac{19}{4}\lambda$$

So

$$i_m = \frac{1}{P}\sum_{n=-M}^{M} a_n e^{-j2\pi\frac{z_m}{\lambda}w_n} = \frac{1}{20}\sum_{n=-5}^{5} a_n e^{-j2\pi\frac{z_m}{\lambda}w_n}$$

$$= \frac{1}{20}\left[a_0 + a_1 e^{-j2\pi\frac{z_m}{\lambda}w_1} + a_{-1}e^{-j2\pi\frac{z_m}{\lambda}w_{-1}} + \cdots\right]$$

$$= \frac{1}{20}\left[a_0 + 2a_1 \cos\left(2\pi\frac{z_m}{\lambda}0.1\right) + \cdots\right]$$

$$= \frac{1}{20}\left[1 + 2\cos\left(0.2\pi\frac{z_m}{\lambda}\right) + 2\cos\left(0.4\pi\frac{z_m}{\lambda}\right) + 2\cos\left(0.6\pi\frac{z_m}{\lambda}\right) + 2\cos\left(0.8\pi\frac{z_m}{\lambda}\right) + 1\cos\left(\pi\frac{z_m}{\lambda}\right)\right]$$

Example : $i_{\pm 1}$

$$i_{\pm 1} = i_m\left(z_m = \pm\frac{1}{4}\lambda\right) = \frac{1}{20}\left[1 + 2\cos\left(0.2\frac{\pi}{4}\right) + 2\cos\left(0.4\frac{\pi}{4}\right) + 2\cos\left(0.6\frac{\pi}{4}\right) + 2\cos\left(0.8\frac{\pi}{4}\right) + 1\cos\left(\frac{\pi}{4}\right)\right]$$

$$= 0.44923$$

10.3-10

$P=10$, $d=\lambda/2$, Woodward-Lawson array, sector pattern ($c=0.5$)

From (10-34) $\quad w_n = n\frac{\lambda}{Pd} = n\frac{\lambda}{10\frac{\lambda}{2}} = 0.2n \qquad |n| \le 5$

The values of w_n and a_n are the same as in Prob. 10.2-6

The element currents follow from (10-35) as

$$i_m = \frac{1}{P}\sum_{n=-M}^{M} a_n e^{-j2\pi\frac{z_m}{\lambda}w_n} = \frac{1}{10}\left[1 + e^{-j2\pi\frac{z_m}{\lambda}0.2} + e^{+j2\pi\frac{z_m}{\lambda}0.2} + e^{-j2\pi\frac{z_m}{\lambda}0.4} + e^{j2\pi\frac{z_m}{\lambda}0.4}\right]$$

$$= \frac{1}{10}\left[1 + 2\cos\left(0.4\pi\frac{z_m}{\lambda}\right) + 2\cos\left(0.8\pi\frac{z_m}{\lambda}\right)\right]$$

And

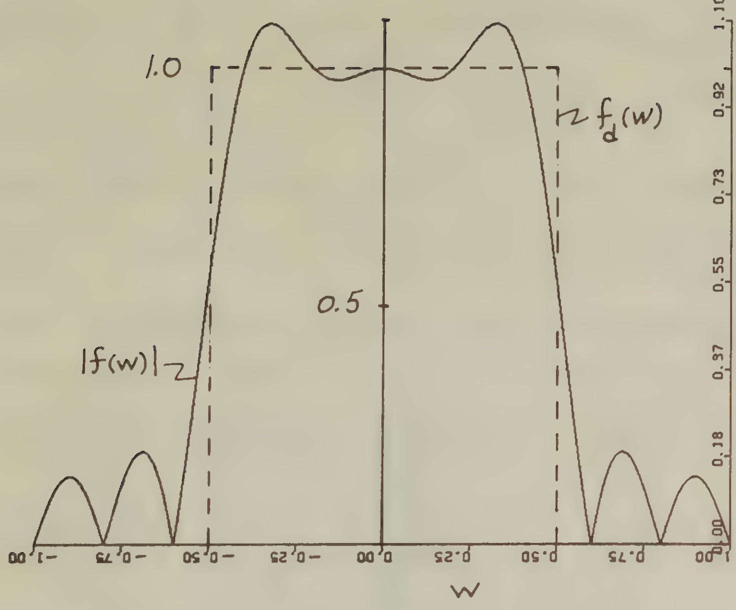
$$z_m = \frac{2m-1}{2}d = \frac{2m-1}{4}\lambda$$

$$z_{-m} = z_m$$

So

I	S(I)	A(I)	PHASE(I)
1	-2.2500	0.071592	0.0
2	-1.7500	-0.079360	0.0
3	-1.2500	-0.100000	0.0
4	-0.7500	0.155754	0.0
5	-0.2500	0.452015	0.0
6	0.2500	0.452015	0.0
7	0.7500	0.155754	0.0
8	1.2500	-0.100000	0.0
9	1.7500	-0.079360	0.0
10	2.2500	0.071592	0.0

10.3-10 (con't)



10.3-11

P=18, d=0.65λ, half-wave dipoles, Woodward-Lawson

$$F_d(\theta)=\begin{cases}1 & 70^\circ<\theta<110^\circ \\ 0 & \text{elsewhere}\end{cases} \qquad g_a(\theta)=\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}$$ collinear $\lambda/2$ dipoles

So

$$f_d(\theta)=\frac{F_d(\theta)}{g_a(\theta)}=\begin{cases}\dfrac{\sin\theta}{\cos(\frac{\pi}{2}\cos\theta)} & 70^\circ<\theta<110^\circ \\ 0 & \text{elsewhere}\end{cases}$$

Sample points are

$$w_n=n\frac{\lambda}{Pd}=n\frac{\lambda}{18(0.65\lambda)}=0.08547n$$

And $a_n=f_d(w_n)$ $\qquad w=\cos\theta$

So

n	$w_n=0.08547n$	$\theta_n=\cos^{-1}w_n$	$a_n=f_d(\theta_n)=\dfrac{\sin\theta_n}{\cos(\frac{\pi}{2}\cos\theta_n)}$
−4	−0.34188	109.9915°	1.09370
−3	−0.25641	104.85	1.05064
−2	−0.17094	99.84	1.02190
−1	−0.08547	94.90	1.00539
0	0	90°	1
1	0.08547	85.097	1.00539
2	0.01709	80.157	1.02190
3	0.2564	75.143	1.05064
4	0.3419	70.009	1.09370

10.3-11 (con't)

$$i_m = \frac{1}{P}\sum_{n=-M}^{M} a_n e^{-j2\pi \frac{z_m}{\lambda} w_n} = \frac{1}{P}\left[1 + 2\sum_{n=1}^{M} a_n \cos\left(2\pi \frac{z_m}{\lambda} w_n\right)\right]$$

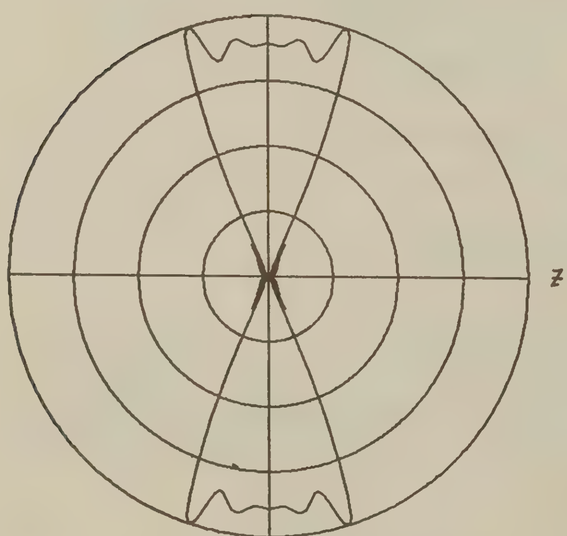
since $a_n = a_{-n}$ & $w_{-n} = -w_n$

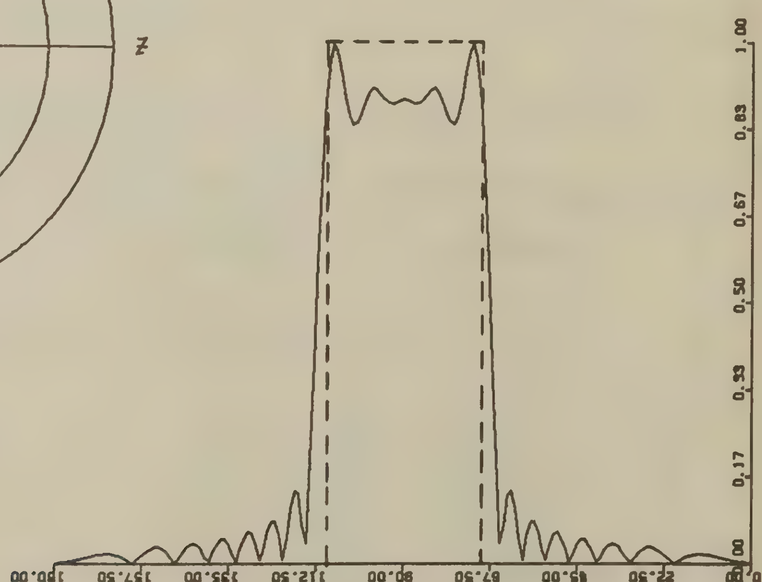
Using $P=18$ and $\{a_n, w_n\}$ from above gives the following

THE ELEMENTS ARE COLLINEAR HALF-WAVE DIPOLES PARALLEL TO THE Z-AXIS

ELEMENT LOCATIONS, CURRENTS, AND PHASES

I	X(I)	Y(I)	Z(I)	A(I)	ALPHA(I)
1	0.0	0.0	-5.5250	0.0443	0.0
2	0.0	0.0	-4.8750	0.0451	179.9999
3	0.0	0.0	-4.2250	0.0490	179.9999
4	0.0	0.0	-3.5750	0.0526	0.0
5	0.0	0.0	-2.9250	0.0635	0.0
6	0.0	0.0	-2.2750	0.0732	179.9999
7	0.0	0.0	-1.6250	0.1076	179.9999
8	0.0	0.0	-0.9750	0.1483	0.0
9	0.0	0.0	-0.3250	0.4665	0.0
10	0.0	0.0	0.3250	0.4665	0.0
11	0.0	0.0	0.9750	0.1483	0.0
12	0.0	0.0	1.6250	0.1076	179.9999
13	0.0	0.0	2.2750	0.0732	179.9999
14	0.0	0.0	2.9250	0.0635	0.0
15	0.0	0.0	3.5750	0.0526	0.0
16	0.0	0.0	4.2250	0.0490	179.9999
17	0.0	0.0	4.8750	0.0451	179.9999
18	0.0	0.0	5.5250	0.0443	0.0





10.3-12

$P=18$, $d=0.65\lambda$, half-wave dipoles (collinear), W-L, csc pattern

$$f_d(\theta)=\frac{F_d(\theta)}{g_a(\theta)}=\begin{cases}\cos 80^\circ \dfrac{\tan\theta}{\cos(\frac{\pi}{2}\cos\theta)} & 0\le\theta\le 80^\circ \\ \dfrac{\sin\theta}{\cos(\frac{\pi}{2}\cos\theta)} & 80^\circ\le\theta\le 90^\circ \\ 0 & \text{elsewhere}\end{cases}$$

Again $w_n = 0.08547n$ $\quad |n|\le M \quad |W_n|\le 1.0$

Then

n	$w_n = 0.08547n$	$\theta_n = \cos^{-1} w_n$	$a_n = f_d(\theta_n)$
0	0	90°	1.0
1	0.08547	85.0969°	1.0054
2	0.17094	80.1575	1.0219
3	0.2564	75.1428	0.7115
4	0.34188	70.0085	0.55551
5	0.42735	64.7005	0.46916
6	0.51282	59.1481	0.41965
7	0.59829	53.2525	0.39421
8	0.68376	46.8618	0.38885
9	0.76923	39.7152	0.40677
10	0.85470	31.2734	0.46614
11	0.94017	19.9199	0.67057

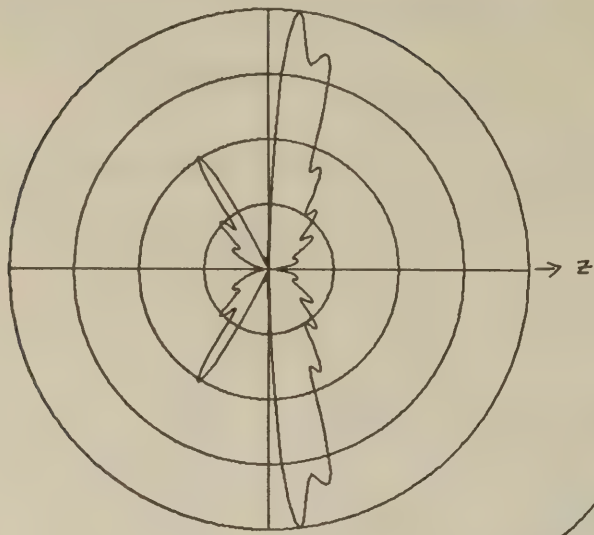
From (10-35

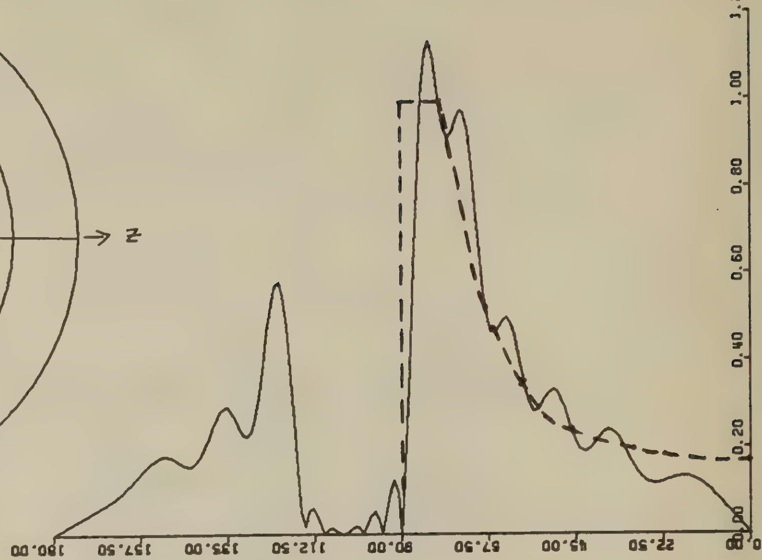
$$i_m=\frac{1}{P}\sum_{n=-M}^{M} a_n e^{-j2\pi\frac{z_m}{\lambda}w_n}=\frac{1}{18}\left[1+\sum_{n=0}^{11} a_n e^{-j2\pi\frac{z_m}{\lambda}w_n}\right]$$

ELEMENT LOCATIONS, CURRENTS, AND PHASES

I	X(I)	Y(I)	Z(I)	A(I)	ALPHA(I)
1	0.0	0.0	-5.5250	0.0478	-163.4000
2	0.0	0.0	-4.8750	0.0105	111.9956
3	0.0	0.0	-4.2250	0.0477	123.6100
4	0.0	0.0	-3.5750	0.0428	-177.2729
5	0.0	0.0	-2.9250	0.0062	160.7680
6	0.0	0.0	-2.2750	0.0811	116.7140
7	0.0	0.0	-1.6250	0.1061	138.5410
8	0.0	0.0	-0.9750	0.1070	66.1159
9	0.0	0.0	-0.3250	0.3381	53.7580
10	0.0	0.0	0.3250	0.3381	-53.7580
11	0.0	0.0	0.9750	0.1070	-66.1159
12	0.0	0.0	1.6250	0.1061	-138.5410
13	0.0	0.0	2.2750	0.0811	-118.7140
14	0.0	0.0	2.9250	0.0062	-160.7680
15	0.0	0.0	3.5750	0.0428	177.2729
16	0.0	0.0	4.2250	0.0477	-123.6100
17	0.0	0.0	4.8750	0.0105	-111.9956
18	0.0	0.0	5.5250	0.0478	163.4000

10.3-12 (con't)





10.4-2

$P = 5$, $d = 0.5\lambda$, Dolph-Chebyshev, -30dB SLL

(a) Following Example 10-5

$$R = 10^{-SLL/20} = 10^{+1.5} = 31.62$$

$$x_o = \cosh\left[\tfrac{1}{P-1}\cosh^{-1} R\right] = \cosh\left[\tfrac{1}{4}\cosh^{-1} 31.62\right] = 1.5872$$

	Normalized current	
$i_2 = \frac{1}{2}x_o^4 = 3.1736$	1.0000	which agrees with Fig. 3-23e
$i_1 = 4i_2 - 2x_o^2 = 7.6557$	2.4123	
$i_o = -2i_2 + 2i_1 + 1 = 9.9641$	3.1397	

(b) From (3-93)

$$D = \frac{\left(\sum_{n=0}^{P-1} A_n\right)^2}{\sum_{n=0}^{P-1} A_n^2} = \frac{[9.9641 + 2(7.6557) + 2(3.1736)]^2}{(9.9641)^2 + 2(7.6557)^2 + 2(3.1736)^2} = \boxed{4.2257}$$

10.4-3

$P = 6$, $d = 0.6\lambda$, Dolph-Chebyshev, -25dB SLL

(a) From (10-43) with $N = 3$

$$f(\psi) = 2\left[i_1 \cos\frac{\psi}{2} + i_2 \cos 3\frac{\psi}{2} + i_3 \cos 5\frac{\psi}{2}\right]$$

10.4-3 (con't)

And

$$\cos 3\frac{\psi}{2} \equiv 4\cos^3\frac{\psi}{2} - 3\cos\frac{\psi}{2}$$

$$\cos 5\frac{\psi}{2} \equiv 16\cos^5\frac{\psi}{2} - 20\cos^3\frac{\psi}{2} + 5\cos\frac{\psi}{2}$$

So

$$f(\psi) = 2\left\{i_1\cos\frac{\psi}{2} + i_2\left(4\cos^3\frac{\psi}{2} - 3\cos\frac{\psi}{2}\right) + i_3\left(16\cos^5\frac{\psi}{2} - 20\cos^3\frac{\psi}{2} + 5\cos\frac{\psi}{2}\right)\right\}$$

$$= 2\left\{(i_1 - 3i_2 + 5i_3)\cos\frac{\psi}{2} + (4i_2 - 20i_3)\cos^3\frac{\psi}{2} + 16i_3\cos^5\frac{\psi}{2}\right\}$$

From (10-41) and (10-40)

$$T_5(x) = 2x\,T_4(x) - T_3(x) = 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x)$$

$$= 16x^5 - 20x^3 + 5x$$

Using $x = x_0\cos\frac{\psi}{2}$ of (10-44) gives

$$T_5(\psi) = 16x_0^5\cos^5\frac{\psi}{2} - 20x_0^3\cos^3\frac{\psi}{2} + 5x_0\cos\frac{\psi}{2}$$

Equating coefficients

$$32i_3 = 16x_0^5 \Rightarrow \quad i_3 = \frac{1}{2}x_0^5$$

$$2(4i_2 - 20i_3) = -20x_0^3 \Rightarrow i_2 = \frac{1}{4}(-10x_0^3 + 20i_3) = \frac{10}{4}(x_0^5 - x_0^3)$$

$$2(i_1 - 3i_2 + 5i_3) = 5x_0 \Rightarrow i_1 = \frac{5}{2}x_0 + 3i_2 - 5i_3 = \frac{5}{2}x_0 + \frac{30}{4}(x_0^5 - x_0^3) - \frac{5}{2}x_0^5$$

Now

$$R = 10^{-SLL/20} = 10^{-(-25/20)} = 10^{1.25} = 17.782$$

From (10-48)

$$x_0 = \cosh\left[\frac{1}{P-1}\cosh^{-1}R\right] = \cosh\left[\frac{1}{5}\cosh^{-1}17.782\right] = 1.266$$

So

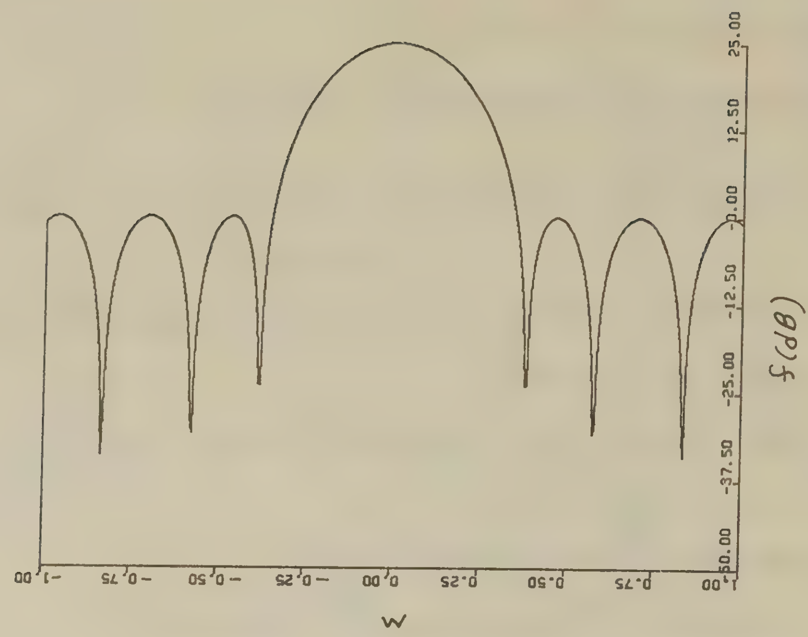
$$i_3 = \frac{1}{2}x_0^5 = \frac{1}{2}(1.266)^5 = \boxed{1.62607}$$

$$i_2 = -\frac{10}{4}x_0^3 + 5i_3 = -\frac{10}{4}(1.266)^3 + 5(1.62607) = \boxed{3.05761}$$

$$i_1 = \frac{5}{2}x_0 + 3i_2 - 5i_3 = \frac{5}{2}(1.266) + 3(3.0576) - 5(1.626) = \boxed{4.20751}$$

THE NUMBER OF ARRAY ELEMENTS = 6

I	S(I)	A(I)	PHASE(I)
1	1.5000	1.626070	0.0
2	0.9000	3.057610	0.0
3	0.3000	4.207510	0.0
4	-0.3000	4.207510	0.0
5	-0.9000	3.057610	0.0
6	-1.5000	1.626070	0.0



<u>10.4-4</u>

The zeros of the ideal Taylor line source are

$$x_n = \pm\sqrt{A^2+(n-\tfrac{1}{2})^2} \qquad (10\text{-}70)$$

Using these to generate the pattern polynomial

$$f_{un}(x) = \prod_{n=1}^{\infty}(x-x_n)(x+x_n) = \prod_{n=1}^{\infty}(x^2-x_n^2) = \boxed{\prod_{n=1}^{\infty}\left[x^2-A^2-(n-\tfrac{1}{2})^2\right]}$$ which is (10-71)

The maximum occurs for $x=0$, so the normalized pattern is

$$f(x) = \frac{f_{un}(x)}{f_{un}(x=0)} = \frac{\prod_{n=1}^{\infty}\left[x^2-A^2-(n-\tfrac{1}{2})^2\right]}{\prod_{n=1}^{\infty}\left[-A^2-(n-\tfrac{1}{2})^2\right]} = \frac{\prod_{n=1}^{\infty}(n-\tfrac{1}{2})^2\left[1+\frac{A^2-x^2}{(n-\frac{1}{2})^2}\right]}{\prod_{n=1}^{\infty}(n-\tfrac{1}{2})^2\left[1+\frac{A^2}{(n-\frac{1}{2})^2}\right]}$$

$$= \frac{\prod_{n=1}^{\infty}\left[1-\frac{x^2-A^2}{(n-\frac{1}{2})^2}\right]}{\prod_{n=1}^{\infty}\left[1+\frac{A^2}{(n-\frac{1}{2})^2}\right]}$$ which is half of (10-72)

Now $\cos \pi y$ has zero locations $y_n = \pm(n-\frac{1}{2})$ $n=1,2,3,\ldots$

So $\cos \pi y = K\prod_{n=1}^{\infty}\left[y^2-(n-\tfrac{1}{2})^2\right] = \prod_{n=1}^{\infty}\left[1-\frac{y^2}{(n-\frac{1}{2})^2}\right]$ normalized

Let $y^2 = x^2-A^2$

$$\cos\left(\pi\sqrt{x^2-A^2}\right) = \prod_{n=1}^{\infty}\left[1-\frac{x^2-A^2}{(n-\frac{1}{2})^2}\right]$$

10.4-4 (con't)

At $x=0$ $\quad \cos(\pi\sqrt{-A^2}) = \cos(j\pi A) = \cosh \pi A = \prod_{n=1}^{\infty}\left[1 + \frac{A^2}{(n-\frac{1}{2})^2}\right]$

Dividing these two equations

$$\frac{\cos[\pi\sqrt{x^2-A^2}]}{\cosh \pi A} = \frac{\prod_{n=1}^{\infty}\left[1 - \frac{x^2-A^2}{(n-\frac{1}{2})^2}\right]}{\prod_{n=1}^{\infty}\left[1 + \frac{A^2}{(n-\frac{1}{2})^2}\right]} = f(x)$$

which is the other half of (10-72)

10.4-5

The zeros of the Taylor line source are

$$x_n = \begin{cases} \pm\sigma\sqrt{A^2+(n-\frac{1}{2})^2} & 1 \le n \le \bar{n} \\ \pm n & \bar{n} \le n \le \infty \end{cases} \qquad (10\text{-}75)$$

The zeros of $\frac{\sin \pi x}{\pi x}$ are $\pm n = \pm 1, \pm 2, \ldots$; so

$$\frac{\sin \pi x}{\pi x} = K \prod_{n=1}^{\infty}(x^2-n^2) = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right) \quad \text{normalized}$$

The Taylor line source pattern is

$$\prod_{n=1}^{\bar{n}-1}\left[1 - \frac{x^2}{\sigma^2(A^2-(n-\frac{1}{2})^2)}\right] \prod_{n=\bar{n}}^{\infty}\left(1 - \frac{x^2}{n^2}\right)$$

$$= \frac{\frac{\sin \pi x}{\pi x}}{\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right)} \prod_{n=1}^{\bar{n}-1}\left[1 - \frac{x^2}{x_n^2}\right] \prod_{n=\bar{n}}^{\infty}\left(1 - \frac{x^2}{n^2}\right) \qquad x_n = \sigma\sqrt{A^2+(n-\tfrac{1}{2})^2}$$

$$= \frac{\sin \pi x}{\pi x} \frac{\prod_{n=1}^{\bar{n}-1}\left[1 - \frac{x^2}{x_n^2}\right]}{\prod_{n=1}^{\bar{n}-1}\left[1 - \frac{x^2}{n^2}\right]} = f(x, A, \bar{n})$$

which is (10-77)

10.4-6

Sampling theorem: $\quad g(t) = \sum_{m=-\infty}^{\infty} g\left(\frac{m}{2B}\right) Sa\left[2\pi B\left(t - \frac{m}{2B}\right)\right]$

Let

$g \to f$; $t \to w$; $\frac{m}{2B} \to w_m^s = \frac{\lambda}{L} m$ or $\frac{1}{2B} \to \frac{\lambda}{L}$ or $B \to \frac{L}{2\lambda}$

and $\frac{L}{2\lambda}$ is the highest "spatial" frequency

Then

$$f(w) = \sum_{m=-\infty}^{\infty} \underbrace{f\left(\frac{\lambda}{L} m\right)}_{a_m} Sa\left[\pi \frac{L}{\lambda}\left(w - \frac{\lambda}{L} m\right)\right] \quad \text{which is (10-80).}$$

10.4-7

From (10-88)
$$\frac{1}{\sqrt{2}} = \frac{1}{R}\cosh\left[\pi\sqrt{A^2-\left(\frac{L}{\lambda}w_{HP}\right)^2}\right]$$

So
$$\cosh^{-1}\frac{R}{\sqrt{2}} = \pi\sqrt{A^2-\left(\frac{L}{\lambda}w_{HP}\right)^2} \quad \text{or} \quad A^2-\left(\frac{L}{\lambda}w_{HP}\right)^2 = \frac{1}{\pi^2}\left[\cosh^{-1}\frac{R}{\sqrt{2}}\right]^2$$
$$w_{HP}^2 = \left(\frac{\lambda}{L}\right)^2\left[A^2-\frac{1}{\pi^2}\left[\cosh^{-1}\frac{R}{\sqrt{2}}\right]^2\right]$$

But $A = \frac{1}{\pi}\cosh^{-1}R$

So
$$w_{HP} = \frac{\lambda}{L}\frac{1}{\pi}\left[(\cosh^{-1}R)^2-\left(\cosh^{-1}\frac{R}{\sqrt{2}}\right)^2\right]^{1/2}$$ which is (10-89).

10.4-8

$$f(n,A,\bar{n}) = \frac{[(\bar{n}-1)!]^2}{(\bar{n}-1+n)!(\bar{n}-1-n)!}\prod_{m=1}^{\bar{n}-1}\left(1-\frac{n^2}{x_m^2}\right) \qquad (10\text{-}87)$$

where
$$x_m^2 = \sigma^2\left(A^2+(m-\tfrac{1}{2})^2\right) \qquad 1 \le m < \bar{n}$$
$$A = \frac{1}{\pi}\cosh^{-1}R \qquad \sigma = \frac{\bar{n}}{\sqrt{A^2+(\bar{n}-\frac{1}{2})^2}}$$

Now $SLL = -25$ dB $\quad R = 17.7828 \quad A = \frac{1}{\pi}\cosh^{-1}R = 1.13655$
$$\sigma^2 = \frac{\bar{n}^2}{A^2+(\bar{n}-\frac{1}{2})^2} = \frac{5^2}{(1.13655)^2+(5-\frac{1}{2})^2} = 1.16054 \qquad \bar{n}=5$$

So

m	$x_m^2 = \sigma^2(A^2+(m-\frac{1}{2})^2)$
1	1.78925
2	4.11033
3	8.75247
4	15.71570

And

n	$\frac{[(\bar{n}-1)!]^2}{(\bar{n}-1+n)!(\bar{n}-1-n)!}$	$\prod_{m=1}^{4}\left(1-\frac{n^2}{x_m^2}\right)$	$f(n,A,\bar{n})$
0	1	1.0	1.0
1	0.8	0.27684	0.221477
2	0.4	−0.013425	−0.005370
3	0.114286	−0.057938	−0.006621
4	0.014286	0.344145	0.004917

10.4-9

TLS, Ex. 10-6, $L=10\lambda$, $R=17.7828$, $\sigma=1.07728$

$$HP_{wi} = \frac{\lambda}{L}\frac{2}{\pi}\left[(\cosh^{-1}R)^2-(\cosh^{-1}\frac{R}{\sqrt{2}})^2\right]^{1/2} = \frac{2}{10\pi}\left[(\cosh^{-1}17.7828)^2-(\cosh^{-1}\frac{17.7828}{\sqrt{2}})^2\right]^{1/2}$$

$$= 0.0978 \quad \checkmark \qquad (10-97)$$

$$HP_i = 2\sin^{-1}\left\{\frac{\lambda}{L\pi}\left[(\cosh^{-1}R)-(\cosh^{-1}\frac{R}{\sqrt{2}})^2\right]^{1/2}\right\}$$

$$= 2\sin^{-1}\left\{\frac{1}{2}HP_{wi}\right\} = 2\sin^{-1}\left\{\frac{0.0978}{2}\right\} = 5.606° \checkmark \ (10-97)$$

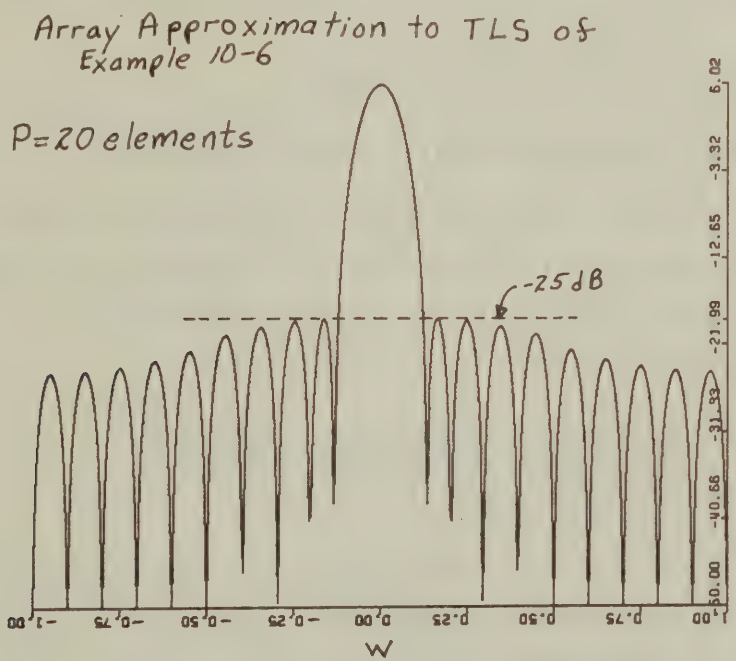
$$HP_w \approx \sigma HP_{wi} = 1.07728\,(0.0978) = 0.10536 \quad \checkmark \quad (10-98)$$

$$HP \approx 2\sin^{-1}\left\{\frac{\lambda\sigma}{L\pi}\left[(\cosh^{-1}R)^2-(\cosh^{-1}\frac{R}{\sqrt{2}})^2\right]^{\frac{1}{2}}\right\}$$

$$= 2\sin^{-1}\left\{\frac{1}{2}HP_w\right\} = 2\sin^{-1}\left\{\frac{0.10536}{2}\right\} = 6.0394° \ \checkmark \ (10-98).$$

10.4-10

(a) Using NEESLAP



(b) The half-power beamwidth of this pattern is $HP_w = 0.105$
This compares to the corresponding Taylor line source of Example 10-6 with $HP_w = 0.1054$.

The side lobes of this array are all below the -25 dB design level but the far-out side lobes do not die off as fast as the Taylor line source of Fig. 10-9.

10.4-11

Taylor line source, $\bar{n}=7$, -30dB SLL, $L=8\lambda$

(a) $R = 10^{-SLL/20} = 10^{1.5} = 31.623$

$$A = \frac{1}{\pi}\cosh^{-1} R = \frac{1}{\pi}\cosh^{-1} 31.623 = 1.320$$

$$\sigma = \frac{\bar{n}}{\sqrt{A^2+(n-\frac{1}{2})^2}} = \frac{7}{\sqrt{(1.320)^2+(6.5)^2}} = 1.05538$$

The sample locations from (10-82) $w_n^s = \frac{\lambda}{L} n = \frac{n}{8}$

The zeros are, from (10-75),

$$x_n = \sigma\sqrt{A^2+(n-\tfrac{1}{2})^2} \qquad n < \bar{n} = 7$$

So

$$x_1 = 1.48969,\ x_2 = 2.10875,\ x_3 = 2.98365,\ x_4 = 3.94780,$$
$$x_5 = 4.94931,\ x_6 = 5.96942$$

The sample values are, from (10-87),

$$a_{\pm n} = f(n, A, \bar{n}) = \frac{[(\bar{n}-1)!]^2}{(\bar{n}-1+n)!(\bar{n}-1-n)!}\prod_{m=1}^{\bar{n}-1}\left(1-\frac{n^2}{x_m^2}\right) \qquad |n|<\bar{n}$$

$$= \frac{[6!]^2}{(6+n)!(6-n)!}\prod_{m=1}^{6}\left(1-\frac{n^2}{x_m^2}\right)$$

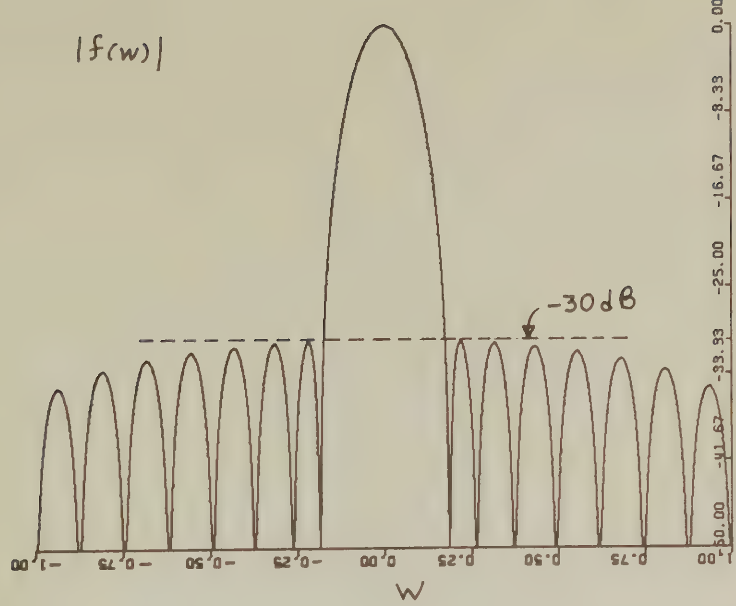
Evaluating and using as input to SPAP,

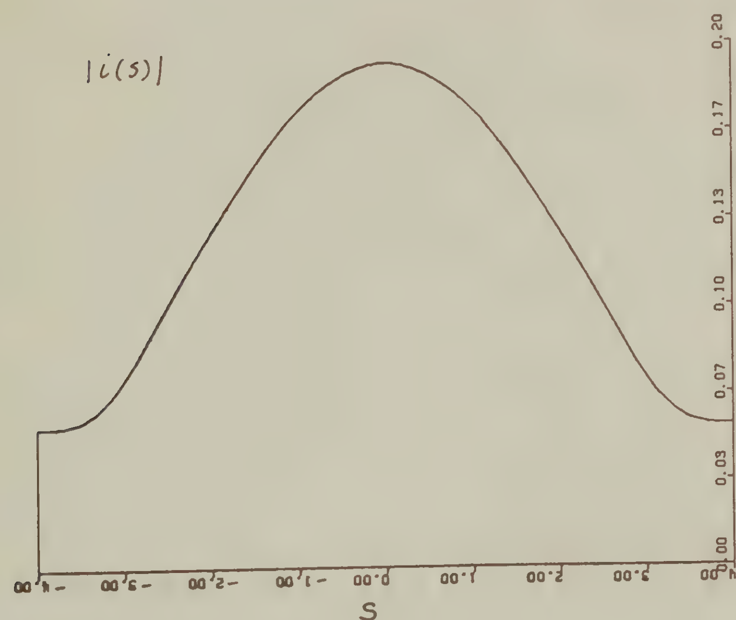
```
SPAP - SAMPLING PATTERN ANTENNA PROGRAM
THIS PATTERN IS EXPRESSIBLE AS A SUM OF SA FUNCTIONS
APERTURE LENGTH= LEN=  8.0000 WAVELENGTHS
NUMBER OF SAMPLE POINTS= NP=   13
```

I	WN(I)	A(I)
1	-0.750000	0.002249
2	-0.625000	-0.004137
3	-0.500000	0.004676
4	-0.375000	-0.001638
5	-0.250000	-0.013133
6	-0.125000	0.282657
7	0.0	1.000000
8	0.125000	0.282657
9	0.250000	-0.013133
10	0.375000	-0.001638
11	0.500000	0.004676
12	0.625000	-0.004137
13	0.750000	0.002249

10.4-11 (con't)

(b)





10.4-12

Taylor line source, −25dB SLL

$A = \frac{1}{\pi}\cosh^{-1} R = 1.13655$ (10-95)

$\sigma = \frac{\bar{n}}{\sqrt{A^2 + (\bar{n} - \frac{1}{2})^2}}$ (10-76) $\quad HP_w \approx \sigma HP_{wi}$

$\bar{n}$	σ
1	0.8054
2	1.063
3	1.0924
4	1.087
5	1.0773
6	1.0683

Initially the HP increases as $\bar{n}$ increases because the high SL's of the $\sin \pi x/\pi x$ pattern are reduced leading to a wider main beam. For larger values of $\bar{n}$, the HP decreases for increasing $\bar{n}$ because the side lobes are raised above what they would be for the $\sin \pi x/\pi x$ pattern.

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